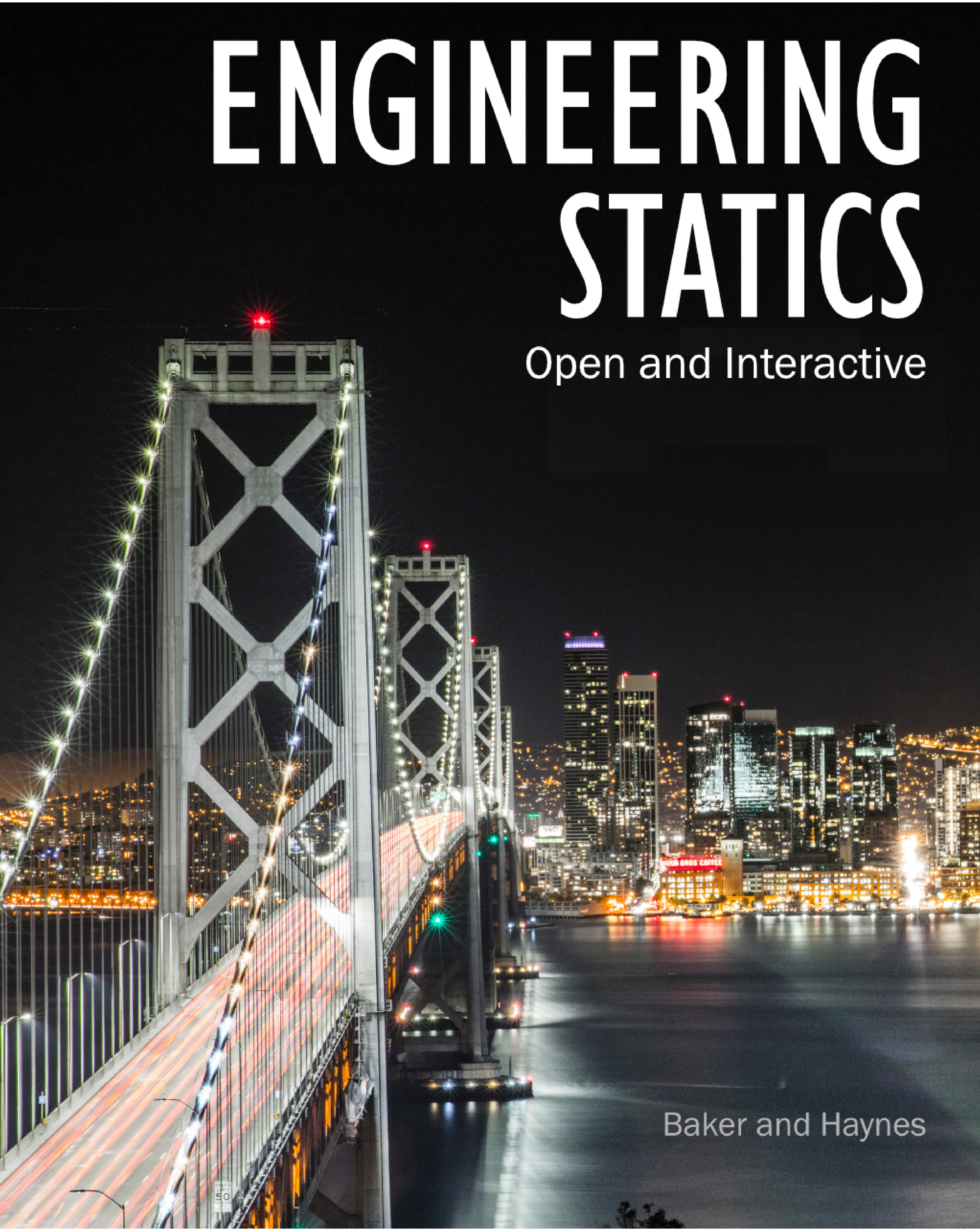
ENGINEERING STATICS

Open and Interactive

Baker and Haynes



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About this Book

Engineering Statics: Open and Interactive is a free, open-source textbook for anyone who wishes to learn more about vectors, forces, moments, static equilibrium, and the properties of shapes. Specifically, it is appropriate as a textbook for Engineering Mechanics: Statics, the first course in the Engineering Mechanics series offered in most university-level engineering programs.

This book's content should prepare you for subsequent classes covering Engineering Mechanics: Dynamics and Mechanics of Materials. At its core, *Engineering Statics* provides the tools to solve static equilibrium problems for rigid bodies. The additional topics of resolving internal loads in rigid bodies and computing area moments of inertia are also included as stepping stones for later courses. We have endeavored to write in an approachable style and provide many questions, examples, and interactives for you to engage with and learn from.

Feedback. Feedback and suggestions can be provided directly to the lead author Dan Baker via email at dan.baker@colostate.edu, by clicking the feedback button in the html footer. When reporting errors, please include a bit of the surrounding text to help locate the problem area in the source. The EngineeringStaticsGoogleGroup is a good place to ask the authors and users questions about the book. Please join the group and say "Hi" if you are using the book for teaching purposes.

Access. The entire book is available for free as an interactive online ebook at https://engineeringstatics.org. While the interactive version works best on larger screens, it will also work smartphones but with some limitations due to limited screen width. A non-interactive PDF version, suitable for printing or offline reading on a tablet or computer, is available at https://engineeringstatics.org/pdf/statics.pdf. The PDF is searchable and easy to navigate using embedded links.

The source files for this book are available on GitHub at https://github.com/dantheboatman/EngineeringStatics.

End of Chapter exercises. The randomized end-of-chapter exercises were made using the Numbas open-source assessment system and Geogebra for the dynamic diagrams. Exercises in the EngineeringStaticsRepository can be freely remixed into your own homework sets or online exams using the NumbasEditor. You can also use the editor to write new questions or modify existing ones.

The exercises can be integrated into your institution's Virtual Learning Environment to set deadlines and automatically record grades. See the NumbasDocumentation for more information. To fully take advantage of all the features, you may need the support of your institution's IT department to install the NumbasLTIprovider.

Please ask questions about Numbas integration and share any good problems you write with the Engineering Statics Group.

On the Cover. Photo of the San Francisco–Oakland Bay Bridge and city skyline, taken from Yerba Buena Island by Artur Westergren.

Image source: https://unsplash.com/photos/Rx92z9dU-mA

Acknowledgements

This book is the vision of a handful of instructors who wanted to create a free and open Engineering Statics textbook filled with dynamic, interactive diagrams to encourage visualization and engineering intuition.

Dr. Baker brought together a team of volunteers from large public universities, small private colleges, and community colleges across the United States to write the text and create the interactive elements. Some content was adapted with permission from Jacob Moore's *Mechanics Map - Open Textbook Project*. http://mechanicsmap.psu.edu/. After two years of development the book was released to the public in 2020.

The book continues to evolve thanks to the contributions, suggestions, and corrections made by users of the text, both professors and students. The original authors are listed below, and others who have contributed are acknowledged in the source code on GitHub.

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Contents

Chapter 1

Introduction to Statics

Engineering Statics is the gateway into *engineering mechanics*, which is the application of Newtonian physics to design and analyze objects, systems, and structures with respect to motion, deformation, and failure. In addition to learning the subject itself, you will also develop skills in the art and practice of problem solving and mathematical modeling, skills that will benefit you throughout your engineering career.

The subject is called "statics" because it is concerned with particles and rigid bodies that are in equilibrium, and these will usually be stationary, i.e. static.

The chapters in this book are:

Introduction to Statics— an overview of statics and an introduction to units and problem solving.

Forces and Other Vectors— basic principles and mathematical operations on force and position vectors.

Equilibrium of Particles— an introduction to equilibrium and problem solving.

Moments and Static Equivalence— the rotational tendency of forces, and simplification of force systems.

Rigid Body Equilibrium— balance of forces and moments for single rigid bodies.

Equilibrium of Structures— balance of forces and moments on interconnected systems of rigid bodies.

Centroids and Centers of Gravity— an important geometric property of shapes and rigid bodies.

Internal Forces— forces and moments within beams and other rigid bodies.

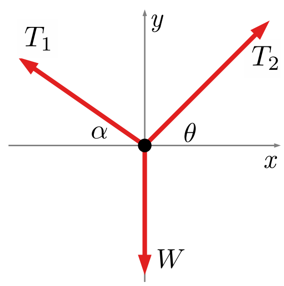
Friction— equilibrium of bodies subject to friction.

Moments of Inertia— an important property of geometric shapes used in many applications.

Your statics course may not cover all of these topics, or may move through them in a different order.

Below are two examples of the types of problems you'll learn to solve in statics. Notice that each can be described with a picture and problem statement, a free-body diagram, and equations of equilibrium.

Equilibrium of a particle: A 140 lb person walks across a slackline stretched between two trees. If angles α and θ are known, find the tension in each end of the slackline.

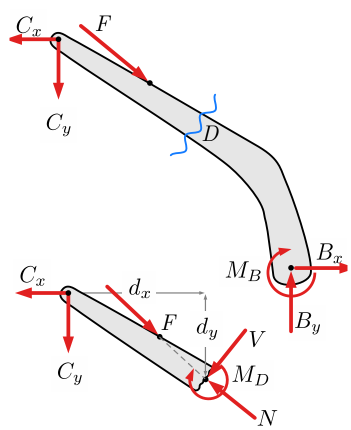


Person's point of contact to slackline:

$$\Sigma F_x = 0$$
$$-T_1 \cos\alpha + T_2 \cos\theta = 0$$

$$\Sigma F_y = 0$$
$$T_1 \sin\alpha + T_2 \sin\theta - W = 0$$

Equilibrium of a rigid body: Given the interaction forces at point C on the upper arm of the excavator, find the internal axial force, shear force, and bending moment at point D.



Section cut FBD:

$$\Sigma F_x = 0$$
$$-C_x + F_x - V_x - N_x = 0$$

$$\Sigma F_y = 0$$
$$-C_y - F_y - V_y + N_y = 0$$

$$\Sigma M_D = 0$$
$$+(d_y)C_x + (d_x)C_y - M_D = 0$$

The knowledge and skills gained in Statics will be used in your other engineering courses, in particular in Dynamics, Mechanics of Solids (also called Strength or Mechanics of Materials), and in Fluid Mechanics. Statics will be a foundation of your engineering career.

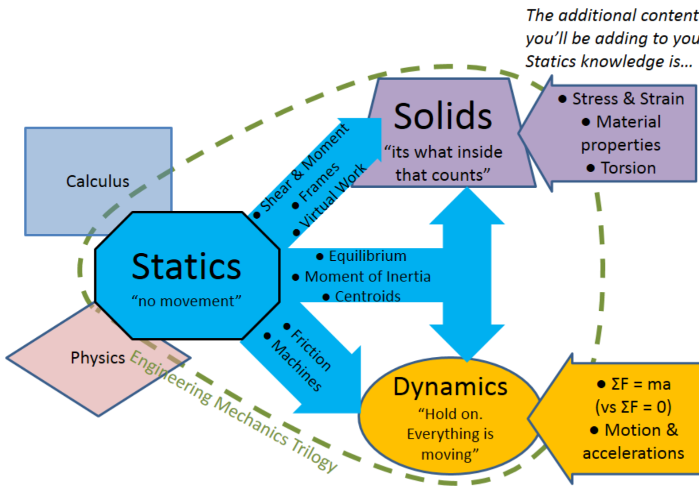


Figure 1.0.1 Map of how Statics builds upon the prerequisites of Calculus and Physics and then informs the later courses of Mechanics of Solids and Dynamics.

1.1 Newton's Laws of Motion

Key Questions

- What are the two types of motion?
- What three relationships do Newton's laws of motion define?
- What are physical examples for each of Newton's three laws of motion?

The English scientist Sir Issac Newton established the foundation of mechanics in 1687 with his three laws of motion, which describe the relation between forces, objects and motion. Motion can be separated into two types:

> **Translation**— where a body changes position without changing its orientation in space, and
>
> **Rotation**— where a body spins about an axis fixed in space, without changing its average position.

Some moving bodies are purely translating, others are purely rotating, and many are doing both. Conveniently, we can usually separate translation and rotation and analyze them individually with independent equations.

Newton's three laws and their implications with respect to translation and rotation are described below.

1.1.1 Newton's 1st Law

Newton's first law states that

> *an object will remain at rest or in uniform motion in a straight line unless acted upon by an external force.*

This law, also sometimes called the "law of inertia," tells us that bodies maintain their current velocity unless a net force is applied to change it. In other words, an object at rest it will remain at rest and a moving object will hold its current speed and direction unless an unbalanced force causes a velocity change. Remember that velocity is a vector quantity that includes both speed and direction, so an unbalanced force may cause an object to speed up, slow down, or change direction.

Figure 1.1.1 This rock is at rest with zero velocity and will remain at rest until a unbalanced force causes it to move.

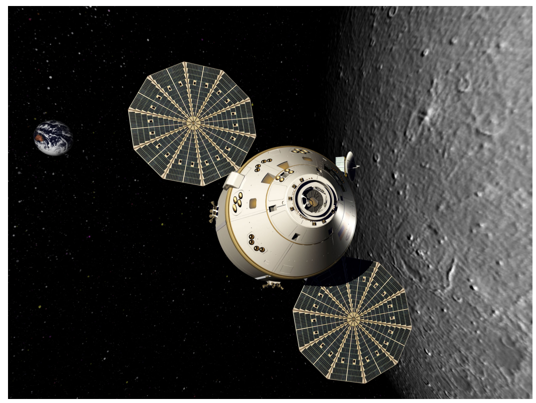


Figure 1.1.2 In deep space, where friction and gravitational forces are negligible, an object moves with constant velocity; near a celestial body gravitational attraction continuously changes its velocity.

Newton's first law also applies to angular velocities, however instead of force, the relevant quantity which causes an object to rotate is called a **torque** by physicists, but usually called a **moment** by engineers. A moment, as you will learn in Chapter 4, is the rotational tendency of a force. Just as a force will cause a change in linear velocity, a moment will cause a change in angular velocity. This can be seen in things like tops, flywheels, stationary bikes, and other objects that spin on an axis when a moment is applied, but eventually stop because of the opposite moment produced by friction.

Figure 1.1.3 A spinning top demonstrates rotary motion.

In the absence of friction this top would spin forever, but the small frictional moment exerted at the point of contact with the table will eventually bring it to a stop.

1.1.2 Newton's 2nd Law

Newton's second law is usually succinctly stated with the familiar equation

$$\mathbf{F} = m\mathbf{a} \tag{1.1.1}$$

where $\mathbf{F}$ is net force, m is mass, and $\mathbf{a}$ is acceleration.

You will notice that the force and the acceleration are in bold face. This means these are vector quantities, having both a magnitude and a direction. Mass on the other hand is a scalar quantity, which has only a magnitude. This equation indicates that a force will cause an object to accelerate in the direction of the net force, and the magnitude of the acceleration will be proportional to the net force but inversely proportional to the mass of the object.

When studying Statics we are only concerned with bodies which are *not* accelerating which simplifies things considerably. When an object is not accelerating $a = 0$, which implies that it is either at rest or moving with a constant velocity. With this restriction Newton's Second Law for translation simplifies to

$$\Sigma\mathbf{F} = 0 \tag{1.1.2}$$

where $\Sigma\mathbf{F}$ is read as "the sum of the forces" and used to indicate the *net* force acting on the object.

Newton's second law for rotational motions is similar

$$\mathbf{M} = I\boldsymbol{\alpha}. \tag{1.1.3}$$

This equation states that a net moment $\mathbf{M}$ acting on an object will cause an angular acceleration $\boldsymbol{\alpha}$ proportional to the net moment and inversely proportional to I, a quantity known as the **mass moment of inertia.** Mass moment of inertia for rotational acceleration is analogous to ordinary mass for linear acceleration. We will have more to say about the moment of inertia in Chapter **??**.

Again, we see that the net moment and angular acceleration are vectors, quantities with magnitude and direction. The mass moment of inertia, on the other hand, is a scalar quantity and has only a magnitude. Also, since Statics deals only with objects which are *not* accelerating $\boldsymbol{\alpha} = 0$, they will always be at rest or rotating with constant angular velocity. With this restriction Newton's second law implies that the net moment on all static objects is zero.

$$\Sigma \mathbf{M} = 0 \tag{1.1.4}$$

1.1.3 Newton's 3rd Law

Newton's Third Law states

> *For every action, there is an equal and opposite reaction.*

The actions and reactions Newton is referring to are **forces**. Forces occur whenever one object interacts with another, either directly like a push or pull, or indirectly like magnetic or gravitational attraction. Any force acting on one body is always paired with another equal-and-opposite force acting on some other body.

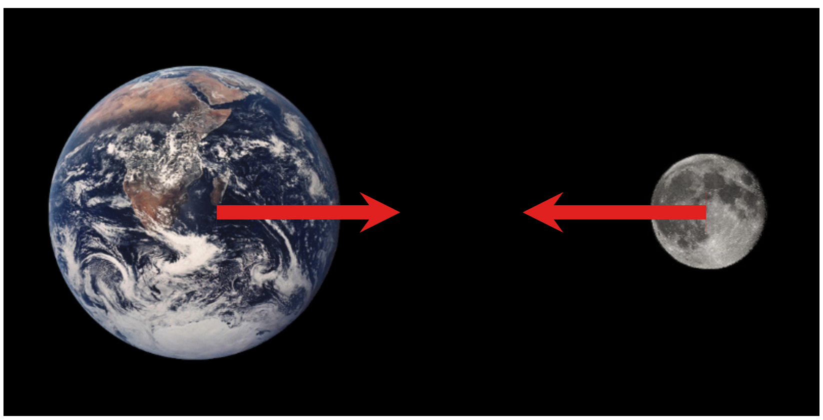


Figure 1.1.4 The earth exerts a gravitational force on the moon, and the moon exerts an equal and opposite force on the earth.

These equal-and-opposite pairs can be confusing, particularly when there are multiple interacting bodies. To clarify, we always begin solving statics problems by drawing a **free-body diagram** — a sketch where we isolate a body or system of interest and identify the forces acting *on* it, while ignoring any forces exerted *by* it on interacting bodies.

Consider the situation in Figure 1.1.5. Diagram (a) shows a book resting on a table supported by the floor. The weights of the book and table are placed at their centers of gravity. To solve for the forces on the legs of the table, we use the free-body diagram in (b) which treats the book and the table as a single system and replaces the floor with the forces of the floor *on* the table. In diagram (c) the book and table are treated as independent objects. By separating them, the equal-and-opposite interaction forces of the book *on* the table and the table *on* the book are exposed.

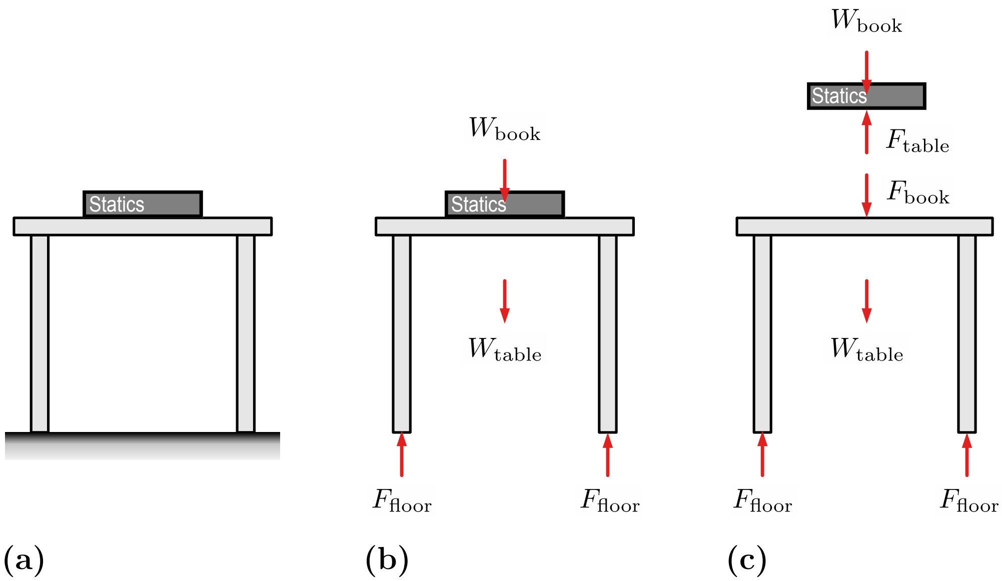


Figure 1.1.5 Free-body diagrams are used to isolate objects and identify relevant forces and moments.

This will be discussed further in Chapter 3 and Chapter 5.

1.2 Units

Key Questions

- What are the similarities and differences between the commonly used unit systems?
- How do you convert a value into different units?
- When a problem mentions the pounds, does this mean pounds-force [lbf] or pounds-mass [lbm]?

Quantities used in engineering usually consist of a numeric value and an associated unit. The value by itself is meaningless. When discussing a quantity you must always include the associated unit, except when the correct unit is 'no units.' The units themselves are established by a coherent **unit system**.

All unit system are based around seven base units, the important ones for Statics being mass, length, and time. All other units of measurement are formed by combinations of the base units. So, for example, acceleration is defined as length $[L]$ divided by time $[t]$ squared, so has units

$$a = [L/t^2].$$

Force is related to mass and acceleration by Newton's second law, so the units of force are

$$F = [mL/t^2].$$

In the United States several different unit systems are commonly used including the SI system, the British Gravitational system, and the English Engineering system.

The SI system, abbreviated from the French *Système International (d'unités)* is the modern form of the metric system. The SI system is the most widely used system of measurement worldwide.

In the SI system, the unit of force is the *newton*, abbreviated N, and the unit of mass is the *kilogram*, abbreviated kg. The base unit of time, used by all systems, is the *second*. Prefixes are added to unit names are used to specify the base-10 multiple of the original unit. One newton is equal to 1 kg $\cdot$ m/s^2 because 1 N of force applied to 1 kg of mass causes the mass to accelerate at a rate of 1 m/s^2.

The British Gravitational system uses the *foot* as the base unit of distance, the second for time, and the slug for mass. Force is a derived unit called the *pound-force*, abbreviated lbf, or pound for short. One pound-force will accelerate a mass of one slug at 1 ft/s^2, so 1 lbf = 1 slug $\cdot$ ft/s^2. On earth, a 1 slug mass weighs 32.174 lbf.

The English Engineering system uses the *pound-mass* as the base unit of mass, where

$$32.174 \text{ lbm} = 1 \text{ slug} = 14.6 \text{ kg}.$$

The acceleration of gravity remains the same as in the British Gravitational system, but a conversion factor is required to maintain unit consistency.

$$1 = \left[\frac{1 \text{ lbf} \cdot \text{s}^2}{32.174 \text{ ft} \cdot \text{lbm}}\right] = \left[\frac{1 \text{ slug}}{32.174 \text{ lbm}}\right] \tag{1.2.1}$$

The advantage of this system is that (on earth) 1 lbm weighs 1 lbf. It is important to understand that mass and weight are not the same thing, however. Mass describes how much matter an object contains, while weight is a force —the effect of gravity on a mass.

You find the weight of an object from its mass by applying Newton's Second Law with the local acceleration of gravity g.

$$W = mg. \tag{1.2.2}$$

Warning 1.2.1

The gravitational "constant" g varies up to about 0.5% across the earth's surface due to factors including latitude and elevation. On the moon, g is about 1.625 m/s^2, and it's nearly zero in outer space.

Don't assume that g always equals 9.81 m/s^2! Always use the correct value of g based on your location and unit system. However, in this course, unless otherwise stated, all objects are located on earth and the values in Table 1.2.2 are applicable.

You can show that 1 lbm mass weighs 1 lbf on earth by first finding the weight with (1.2.2) with $g = 32.174$ ft/s^2, then applying the conversion factor (1.2.1).

$$\begin{aligned} W &= mg \\ &= (1 \text{ lbm})(32.174 \text{ ft/s}^2) \\ &= \left(\cancel{32.174}\ \frac{\cancel{\text{lbm}}\cancel{\text{ft}}}{\cancel{\text{s}^2}}\right)\left[\frac{1 \text{ lbf}\cdot\cancel{\text{s}^2}}{\cancel{32.174}\ \cancel{\text{ft}}\cdot\cancel{\text{lbm}}}\right] \\ &= 1 \text{ lbf} \end{aligned}$$

Table 1.2.2 shows the standard units of weight, mass, length, time, and gravitational acceleration in three unit systems.

Table 1.2.2 Fundamental Units

Unit System	Force	Mass	Length	Time	g (Earth)
SI	N	kg	m	s	9.81 m/s^2
British Gravitational	lbf	slug	ft	s	32.174 ft/s^2
English Engineering	lbf	lbm	ft	s	1 lbf/1 lbm

Example 1.2.3

How much does a 5 kg bag of flour weigh?

Hint. A value in kg is a mass. Weight is a force.

Solution.

$$\begin{aligned} W &= mg \\ &= 5 \text{ kg}(9.81 \text{ m/s}^2) \\ &= 49.05 \text{ N} \end{aligned}$$

Example 1.2.4

How much does a 5 lb bag of sugar weigh?

Hint. When someone says “pounds” they probably mean “pounds-force”.

Solution. The weight was given:

$$w = 5 \text{ lb} = 5 \text{ lbf}.$$

On earth, the mass is 5 lbm, or

$$m = 5 \text{ lbm}\left[\frac{\text{slug}}{32.174 \text{ lbm}}\right] = 0.155 \text{ slug}$$

using the conversion fractor in (1.2.1).

Thinking Deeper 1.2.5 Does 1 pound-mass equal 1 pound-force?

Of course not; they have completely different units!
Although a 1 lb mass weighs 1 lb on earth, pounds-mass and pounds-force are not *equal*. If you take a 1 lbm mass to the moon, its mass doesn't change, but it weighs significantly less than it does on earth. The same mass in deep space is weightless!

Awareness of units will help you prevent errors in your engineering calculations. You should always:

- Pay attention to the units of every quantity in the problem. Forces should have force units, distances should have distance units, etc.
- Use the unit system given in the problem statement.
- Avoid unit conversions when possible. If you must, convert given values to a consistent set of units and stick with them.
- Check your work for unit consistency. You can only add or subtract quantities which have the same units. When multiplying or dividing quantities with units, multiply or divide the units as well. The units of quantities on both sides of the equals sign must be the same.
- Develop a sense of the magnitudes of the units and consider your answers for reasonableness. A kilogram is about 2.2 times as massive as a pound-mass and a newton weighs about a quarter pound.
- Be sure to include units with every answer.

1.3 Forces

Key Questions

- What are some of the fundamental types of forces used in statics?
- Why do we often simplify distributed forces with equivalent forces?

Statics is a course about forces and we will have a lot to say about them. At its simplest, a force is a "push or pull," but forces come from a variety of sources and occur in many different situations. As such we need a specialized vocabulary to talk about them. We are also interested in forces that cause rotation, and we have special terms to describe these too.

Some terms used to describe forces are given below; others will be defined as needed later in the book.

Point Forces, also called **concentrated forces**, are forces that act at a single point. Examples are the push you give to open a door, the thrust of a rocket engine, or the pull a the chain suspending a wrecking ball. Point forces are actually an idealization, because real forces always act over an area and not at a mathematical point. However, point forces are the easiest type to deal with computationally so we will usually represent other types of forces as equivalent concentrated forces.

Distributed forces are forces that are spread out over a line, area or volume. Steam pressure in a boiler and the weight of snow on a roof are examples of forces distributed over an area. Distributed forces are represented graphically by an array of force vectors.

Body forces are distributed forces acting over the volume of a body. The most common body force is the body's weight, but there are others including buoyancy and forces caused by electric and magnetic fields. Weight and buoyancy will be the only body forces we consider in this book.

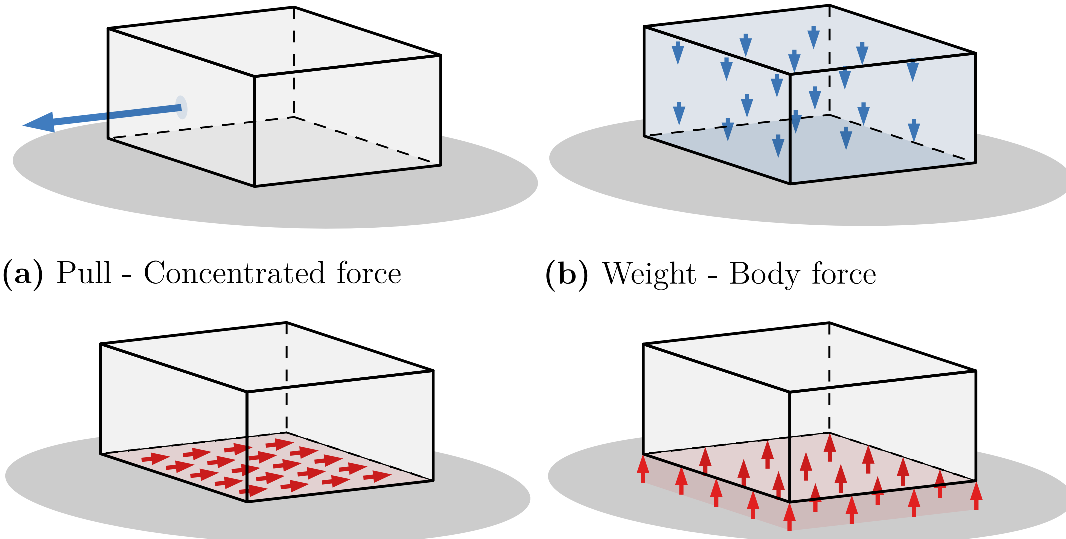
In many situations, body forces are small in comparison to the other forces acting on the object, and as such may be neglected. In practice, the decision to neglect forces must be made on the basis of sound engineering judgment; however, in this course you should consider the weight in your analysis if the problem statement provides enough information to determine it, otherwise you may ignore it.

Loads are the forces which an object must support in order to perform its function. Loads can be either static or dynamic, however only static loads will be considered here. Forces which hold a loaded object in equilibrium or hold parts of an object together are not considered loads.

Reaction forces or simply **reactions** are the forces and moments which hold or constrain an object or mechanical system in equilibrium. They are called the reactions because they react when other forces on the system change. If the load on a system increases, the reaction forces will automatically increase in response to maintain equilibrium. Reaction forces are introduced in Chapter 3 and reaction moments are introduced in Chapter 5.

Internal forces are forces which hold the parts an object or system together. Internal forces will be discussed in Chapter **??**.

As an example of the various types of forces, consider a heavy crate being pulled by a rope across a rough surface.



(a) Pull - Concentrated force

(b) Weight - Body force

(c) Friction - Distributed force

(d) Normal Force - Distributed force

Figure 1.3.1 Forces on a crate being pulled across a rough surface.

The pull of the rope and the weight of the crate are *loads*. The rope applies a force at a single point, so is a *concentrated force*. The force of the ground holding the crate in equilibrium is a *reaction force*. This force can be divided into two components: a **tangential** friction component which acts parallel to the ground and resists the pull of the cable, and a **normal** component which acts perpendicular to the bottom surface and supports the crate's weight. The normal and tangential components are *distributed forces* since they act over the bottom surface area. The weight is also a distributed force, but one that acts over the entire crate so it's considered a *body force*. For computational simplicity we usually model all these distributed forces as equivalent concentrated forces. This process is discussed in Chapter 7.

1.4 Problem Solving

Key Questions

- What are some strategies to practice selecting a tool from your problem-solving toolbox?
- What is the basic problem-solving process for equilibrium?

Statics may be the first course you take where you are required to decide on your own how to approach a problem. Unlike your previous physics courses, you can't just memorize a formula and plug-and-chug to get an answer; there are often multiple ways to solve a problem, not all of them equally easy, so before you begin you need a plan or strategy. This seems to cause a lot of students difficulty.

The ways to think about forces, moments and equilibrium, and the mathematics used to manipulate them are like tools in your toolbox. Solving statics problems requires acquiring, choosing, and using these tools. Some problems can be solved with a single tool, while others require multiple tools. Sometimes one tool is a better choice, sometimes another. You need familiarity and practice to get skilled using your tools. As your skills and understanding improve, it gets easier to recognize the most efficient way to get a job done.

Struggling statics students often say things like:

> "I don't know where to start the problem."
>
> "It looks so easy when you do it."
>
> "If I only knew which equation to apply, I could solve the problem."

These statements indicate that the students think they know how to use their tools, but are skipping the planning step. They jump right to writing equations and solving for things without making much progress towards the answer, or they start solving the problem using a reasonable approach but abandon it in mid-stream to try something else. They get lost, confused and give up.

Choosing a strategy gets easier with experience. Unfortunately, the way you get that experience is to solve problems. It seems like a chicken and egg problem and it is, but there are ways around it. Here are some suggestions which will help you become a better problem-solver.

- Get fluent with the math skills from algebra and trigonometry.
- Do lots of problems, starting with simple ones to build your skills.
- Study worked out solutions, however don't assume that just because you understand how someone else solved a problem that you can do it yourself without help.
- Solve problems using multiple approaches. Confirm that alternate approaches produce the same results, and try to understand why one method was easier than the other.
- Draw neat, clear, labeled diagrams.
- Familiarize yourself with the application, assumptions, and terminology of the methods covered in class and the textbook.
- When confused, identify what is confusing you and ask questions.

The majority of the topics in this book focus on equilibrium. The remaining topics are either preparing you for solving equilibrium problems or setting you up with skills that you will use in later classes. For equilibrium problems, the problem-solving steps are:

1. Read and understand the problem.

2. Identify what you are asked to find and what is given.
3. Stop, think, and decide on an strategy.
4. Draw a free-body diagram and define variables.
5. Apply the strategy to solve for unknowns and check solutions.
6. (a) Write equations of equilibrium based on the free-body diagram.
   (b) Check if the number of equations equals the number of unknowns. If it doesn't, you are missing something. You may need additional free-body diagrams or other relationships.
   (c) Solve for unknowns.
7. Conceptually check solutions.

Using these steps does not guarantee that you will get the right solution, but it will help you be critical and conscious of your chosen strategies. This reflection will help you learn more quickly and increase the odds that you choose the right tool for the job.

Chapter 2

Forces and Other Vectors

Before you can solve statics problems, you will need to understand the basic physical quantities used in Statics: scalars and vectors.

Scalars are physical quantities that have no associated direction and can be described by a positive or negative number, or even zero. Scalar quantities follow the usual laws of algebra, and most scalar quantities have units. Mass, time, temperature, and length are all scalars.

Vectors represent physical quantities that have magnitude and direction. Vectors are identified by a symbolic name which will be typeset in bold like **r** or **F** to indicate its vector nature. The primary vector quantity you will encounter in statics will be **force**, but **moment** and **position** are also important vectors. Computations involving vectors must always consider the directionality of each term and follow the rules of vector algebra as described in this chapter.

2.1 Vectors

Key Questions

- How is a vector different than a scalar?
- How do you identify the tip, tail, line of action, direction, and magnitude of any drawn vector?
- What are the standard notations for vectors and scalars in this textbook?
- What is the difference between the sense and orientation of a vector?

You can visualize a vector as an **arrow** pointing in a particular direction. The **tip** is the pointed end and the **tail** the trailing end. The tip and tail of a vector define a **line of action**. A line of action can be thought of as an invisible string along which a vector can slide. Sliding a vector along its line of action does not change its magnitude or its direction. Sliding a vector can be a handy way to simplify vector problems.

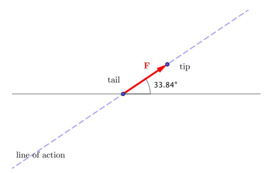


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Figure 2.1.1 Vector Definitions

The standard notation for a vector uses the vector's name in bold font, or an arrow or bar above the vector's name. All three of these notations mean the same thing.

$$\mathbf{F} = \vec{F} = \bar{F} = \text{ a vector named } F$$

Most printed works including this book will use the bold symbol to indicate vectors, but for handwritten work, you and your instructor will use the bar or arrow notation.

Force vectors acting on physical objects have a **point of application**, which is the point where the force is applied. Other vectors, such as moment vectors, are **free vectors**, which means that the point of application is not significant. Free vectors can be moved freely to any location as long as the magnitude and direction are maintained.

The vector's **magnitude** is a *positive real number* including units which describes the 'strength' or 'intensity' of the vector. Graphically a vector's magnitude is represented by the length of its vector arrow, and symbolically by enclosing the vector's symbol with vertical bars. This is the same notation as for the absolute value of a number. The absolute value of a number and the magnitude of a vector can both be thought of as a distance from the origin, so the notation is appropriate. By convention the magnitude of a vector is also indicated by the same letter as the vector, but in non-bold font.

$$F = |\mathbf{F}| = \text{ the magnitude of vector } \mathbf{F}$$

By itself, a vector's magnitude is a scalar quantity, but it makes no sense to speak of a vector with a negative magnitude so vector magnitudes are always positive or zero. Multiplying a vector by -1 produces a vector with the same magnitude but pointing in the opposite direction.

Vector directions are described with respect to a **coordinate system**. A coordinate system is an arbitrary reference system used to establish the **origin** and the primary directions. Distances are usually measured from the origin, and directions from a primary or **reference direction**. You are probably familiar with the Cartesian coordinate system with mutually perpendicular x, y and z axes and the origin at their intersection point.

Another way of describing a vector's direction is to specify its **orientation** and **sense**. Orientation is the angle the vector's line of action makes with a specified reference direction, and sense defines the direction the vector points along its line of action. A vector with a positive sense points towards the positive end of the reference axis and vice-versa. A vector representing an object's weight has a vertical reference direction and a downward sense or negative sense, for example.

A third way to represent a vector is with a **unit vector** multiplied by a **scalar component**. Unit vectors are vectors with a magnitude of one (unitless), and scalar components are signed values with units. Together, they fully define a vector quantity; the unit vector specifies the direction of its line of action, and the scalar component specifies its magnitude and sense. The scalar component "scales" the unit vector.

Be careful not to confuse scalar components, which can be positive or negative, with vector magnitudes, which are always positive.

Vectors are either constant or vary as a function of time, position, or something else. For example, if a force varies with time according to the function $F(t) = (10\text{ N/s})t$, where t is the time in seconds, then the force will be 0 N at $t = 0$ s and increase by 10 N each second thereafter.

2.2 One-Dimensional Vectors

Key Questions

- Given two one-dimensional vectors, how do you compute and then draw the resultant?
- What happens when you multiply a vector by a scalar?

The simplest vector calculations involve one-dimensional vectors. You can learn some important terminology here without much mathematical difficulty. In one-dimensional situations, all vectors share the same line of action, but may point towards either end. If the line of action has a positive end like a coordinate axis does, then a vector pointing towards that end will have a positive scalar component.

2.2.1 Vector Addition

Adding multiple vectors together finds the **resultant** vector. Resultant vectors can be thought of as the *sum* of or *combination* of two or more vectors.

To find the resultant vector $\mathbf{R}$ of two one-dimensional vectors $\mathbf{A}$ and $\mathbf{B}$ you can use the tip-to-tail technique in Figure 2.1.1 below. In the tip-to-tail technique, you slide vector $\mathbf{B}$ until its tail is at the tip of $\mathbf{A}$, and the vector from the tail of **A** to the tip of **B** is the resultant **R**. Note that vector addition is commutative: the resultant **R** is the same whether you add **A** to **B** or B to A.

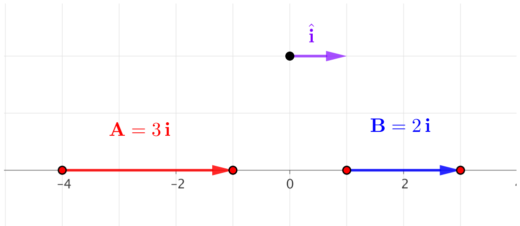

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Figure 2.2.1 One Dimensional Vector Addition

2.2.2 Vector Subtraction

The easiest way to handle vector subtraction is to add the negative of the vector you are subtracting to the other vector. In this way, you can still use the tip-to-tail technique after flipping the vector you are subtracting.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{2.2.1}$$

Example 2.2.2 Vector subtraction.

Find $\mathbf{A} - \mathbf{B}$ where $\mathbf{A} = 2\ \mathbf{i}$ and $\mathbf{B} = 3\ \mathbf{i}$.

Solution. You can simulate this in Figure 2.2.1.

1. Set **A** to a value of $2\ \mathbf{i}$ and **B** to a value of $-3\ \mathbf{i}$, the negative of its actual value.
2. Move the vectors until they are tip-to-tail. The order does not matter because vector addition is commutative.

$$\mathbf{R} = -1\ \mathbf{i}.$$

2.2.3 Vector Multiplication by a Scalar

Multiplying or dividing a vector by a scalar changes the vector's magnitude but maintains its original line of action. One common transformation is to find the negative of a vector. To find the negative of vector **A**, we multiply it by -1; in equation form

$$-\mathbf{A} = (-1)\mathbf{A}$$

Spatially, the effect of negating a vector this way is to rotate it by 180°. The magnitude, line of action, and orientation stay the same, but the sense reverses so now the arrowhead points in the opposite direction.

2.3 2D Coordinate Systems & Vectors

Key Questions

- Why are orthogonal coordinate systems useful?
- How do you transform between polar and Cartesian coordinates?

A coordinate system gives us a frame of reference to describe a system that we would like to analyze. In statics we normally use **orthogonal** coordinate systems, where orthogonal means "perpendicular." In an orthogonal coordinate system the coordinate direction are perpendicular to each other and thereby independent. The intersection of the coordinate axes is called the **origin**, and measurements are made from there. Both points and vectors are described with a set of numbers called the **coordinates**. For points in space, the coordinates specify the distance you must travel in each of the coordinate directions to get from the origin to the point in question. Together, the coordinates can be thought of as specifying a **position vector**, a vector from the origin directly to the point. The position vector gives the magnitude and direction needed to travel directly from the origin to the point.

In the case of force vectors, the coordinates are the **scalar components** of the force in each of the coordinate directions. These components locate the tip of the vector and they can be interpreted as the fraction of the total force which acts in each of the coordinate directions.

Three coordinate directions are needed to map our real three-dimensional world but in this section we will start with two, simpler, two-dimensional orthogonal systems: **rectangular** and **polar** coordinates**,** and the tools to convert from one to the other.

2.3.1 Rectangular Coordinates

The most important coordinate system is the **Cartesian** system, which was named after the French mathematician René Descartes. In two dimensions the coordinate axes are straight lines rotated 90° apart named x, and y.

In most cases, the x axis is horizontal and points to the right, and the y axis points vertically upward, however, we are free to rotate or translate this entire coordinate system if we like. It is usually mathematically advantageous to establish the origin at a convenient point to make measurements from, and to align one of the coordinate axes with a major feature of the problem.

Points are specified as an ordered pair of coordinate values separated by a *comma* and enclosed in parentheses, $P = (x, y)$.

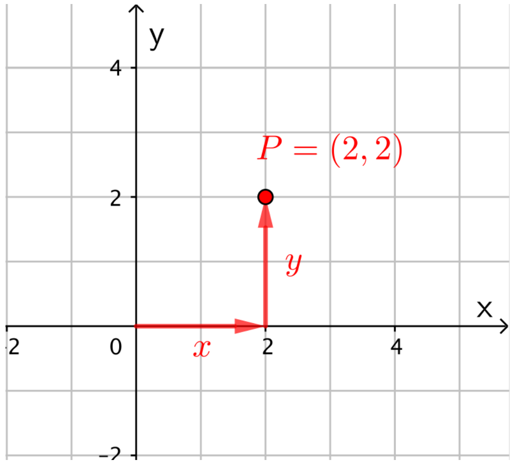

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Figure 2.3.1 Cartesian Coordinate System

Similarly, forces and other vectors will be specified with an ordered pair of scalar components enclosed by angle brackets,

$$\mathbf{F} = \langle F_x, F_y \rangle.$$

2.3.2 Polar Coordinates

The **polar** coordinate system is an alternate orthogonal system which is useful in some situations. In this system, a point is specified by giving its distance from the origin r, and θ, an angle measured counter-clockwise from a reference direction – usually the positive x axis.

In this text, points in polar coordinates will be specified as an ordered pair of values separated by a *semicolon* and enclosed in parentheses

$$P = (r\ ; \theta).$$

Angles can be measured in either radians or degrees, so be sure to include a degree sign on angle θ if that is what you intend.

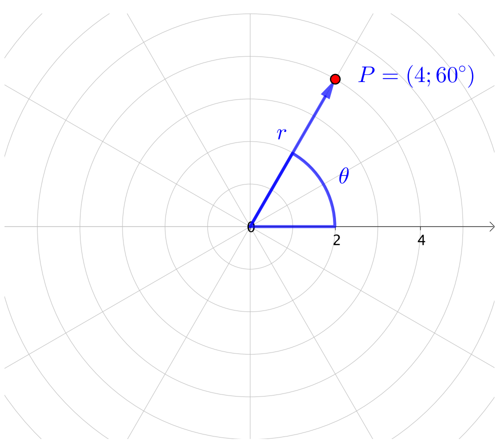

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Figure 2.3.2 Polar Coordinate System

2.3.3 Coordinate Transformation

You should be able to translate points from one coordinate system to the other whenever necessary. The relation between (x, y) coordinates and $(r; \theta)$ coordinates are illustrated in the diagram and right-triangle trigonometry is all that is needed to convert from one representation to the other.

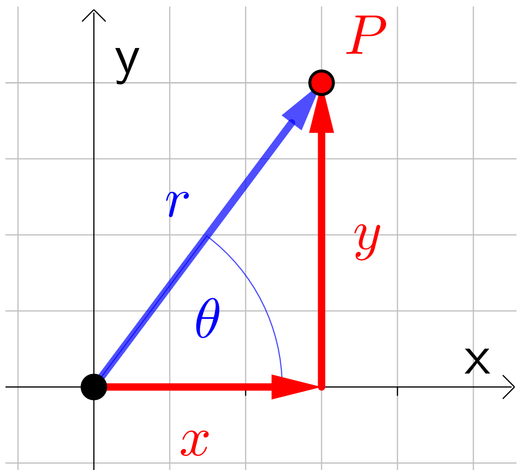

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Figure 2.3.3 Coordinate Transformation

Rectangular To Polar for points (Given: x and, y).

$$r = \sqrt{x^2 + y^2} \tag{2.3.1}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{2.3.2}$$

$$P = (r\ ; \theta) \tag{2.3.3}$$

Note 2.3.4

Take care when using the inverse tangent function on your calculator. Calculator angles are always in the first or fourth quadrant, and you may need to add or subtract 180° to the calculator angle to locate the point in the correct quadrant.

Polar to Rectangular for points (Given: r and, θ).

$$x = r\cos\theta \tag{2.3.4}$$

$$y = r\sin\theta \tag{2.3.5}$$

$$P = (x, y) \tag{2.3.6}$$

Rectangular To Polar for forces (Given: rectangular components). If you are working with forces rather than distances, the process is exactly the same but triangle is labeled differently. The hypotenuse of the triangle is the magnitude of the vector, and sides of the right triangle are the scalar components of the force, so for vector **A**

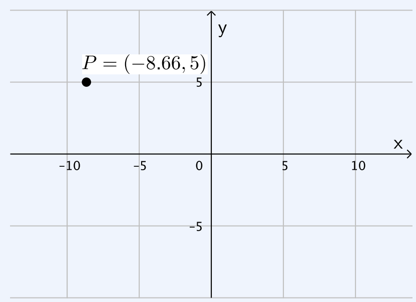
$$A = \sqrt{A_x^2 + A_y^2} \tag{2.3.7}$$
$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{2.3.8}$$
$$\mathbf{A} = (A\,;\theta) \tag{2.3.9}$$

Polar to Rectangular for forces (Given: magnitude and direction).

$$A_x = A\cos\theta \tag{2.3.10}$$
$$A_y = A\sin\theta \tag{2.3.11}$$
$$\mathbf{A} = \langle A_x, A_y\rangle = A\langle\cos\theta, \sin\theta\rangle \tag{2.3.12}$$

Example 2.3.5 Rectangular to Polar Representation.

Express point $P = (-8.66, 5)$ in polar coordinates.



Solution 1. Given: $x = -8.66$, $y = 5$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-8.66)^2 + (5)^2} \\ &= 10 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{5}{-8.66}\right) \\ &= \tan^{-1}(-0.577) \\ &= -30^\circ \end{aligned}$$

You must be careful here and use some common sense. The -30° angle your calculator gives you in this problem is incorrect because point P is in the second quadrant, but your calculator doesn't know this. It can't tell whether the argument of $\tan^{-1}(-0.577)$ is negative because the x was negative or because the y was negative, so it must make an assumption and in this case it is wrong.

The arctan function on calculators will always return values in the first and fourth quadrant. If, by inspection of the x and the y coordinates, you see that the point is in the second or third quadrant, you must add or subtract 180° to the calculator's answer.

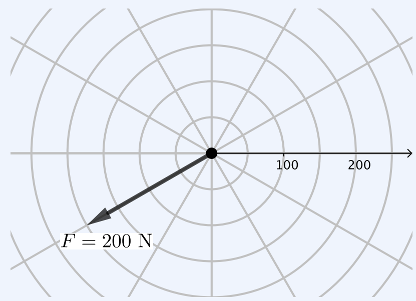
So in this problem, θ is really $-30° + 180°$. After making this adjustment, the location of P in polar coordinates is:

$$P = (10; 150°)$$

Solution 2. Most scientific calculators include handy polar-to-rectangular and rectangular-to-polar functions that can save you time and help you avoid errors. Perhaps you should google your calculator model[1] to find out if yours does and learn how to use it?

Example 2.3.6 Polar to Rectangular Representation.

Express 200 N force **F** as a pair of scalar components.



Solution 1. Given: The magnitude of force $\mathbf{F} = 200$ N, and from the diagram we see that the direction of **F** is 30° counter-clockwise from the negative x axis.
Letting $\theta = 30°$ we can find the components of **F** with right triangle trigonometry.

$$\begin{aligned} F_x &= F\cos\theta \\ &= 200 \text{ N} \cos 30° \\ &= 173.2 \text{ N} \end{aligned} \qquad \begin{aligned} F_y &= F\sin\theta \\ &= 200 \text{ N} \sin 30° \\ &= 100 \text{ N} \end{aligned}$$

Since the force points down and to the left into the third quadrant, these values are actually negative, and the signs must be applied manually.
After making this adjustment, the location of **F** expressed in rectangular coordinates is:

$$\mathbf{F} = \langle -173.2 \text{ N}, -100 \text{ N} \rangle$$

Solution 2. If you would prefer not to apply the negative signs by hand, you can convert the 30° to an angle measured from the positive x axis and let your calculator takes care of the signs. You may use either $\theta = 30° \pm 180°$.

[1] `google.com`

For $\theta = -150°$

$$F_x = F\cos\theta \qquad\qquad F_y = F\sin\theta$$
$$= 200 \text{ N}\cos(-150°) \qquad\qquad = 200 \text{ N}\sin(-150°)$$
$$= -173.2 \text{ N} \qquad\qquad = -100 \text{ N}$$

$$\mathbf{F} = \langle -173.2 \text{ N}, -100 \text{ N}\rangle$$

Although this approach is mathematically correct, experience has shown that it can lead to errors and we recommend that when you work with right triangles, use angles between zero and 90°, and apply signs manually as required by the physical situation.

2.4 3D Coordinate Systems & Vectors

Key Questions

- What is a right-hand Cartesian coordinate system?
- What are direction cosine angles and why are they always less than 180°?
- How are spherical coordinates different than cylindrical coordinates?

In this section we will discuss four methods to specify points and vectors in three-dimensional space.

The most commonly used method is an extension of two-dimensional **rectangular coordinates** to three-dimensions. Alternately, points and vectors in three dimensions can be specified in terms of **direction cosines**, or using **spherical** or **cylindrical** coordinate systems. These will be discussed in the following sections.

You will often need to convert from one representation to another. Good visualization skills are helpful here.

2.4.1 Rectangular Coordinates

We can extend the two-dimensional Cartesian coordinate system into three dimensions easily by adding a z axis perpendicular to the two-dimensional Cartesian plane. The notation is similar to the notation used for two-dimensional vectors. Points and forces are expressed as ordered triples of rectangular coordinates following the same notation used previously.

$$P = (x, y, z) \qquad\qquad \mathbf{F} = \langle F_x, F_y, F_z \rangle$$

For nearly all three-dimensional problems, you will need the rectangular x, y, and z locations of points in space and components of vectors before proceeding with the computations. If you are given the components upfront, then you are set to move forward, but otherwise, you will need to transform one coordinate system into rectangular coordinates.

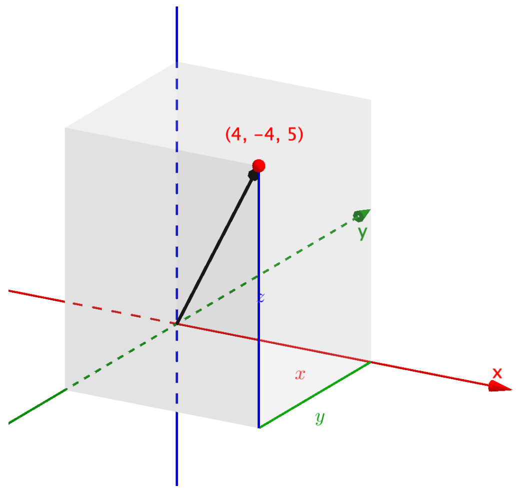


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Figure 2.4.1 Three-Dimensional Rectangular Coordinates

Thinking Deeper 2.4.2 Right Handed Coordinate Systems.

Does it matter which way the axes are oriented? Is it OK to make the x axis point left or the y axis point down?

In one sense, it doesn't matter at all. The positive directions of the coordinate axes are arbitrary. On the other hand, it's convenient for everyone if we agree on a standard orientation. In mathematics and engineering the default is a **right-handed coordinate system**, where the coordinate axes are oriented according to the **right hand rule** shown in the figure.

To apply the right-hand rule, orient your thumb and first two fingers at right angles to each other and align them with three coordinate axes. Starting with your thumb, name your the axes in alphabetical order x-y-z.

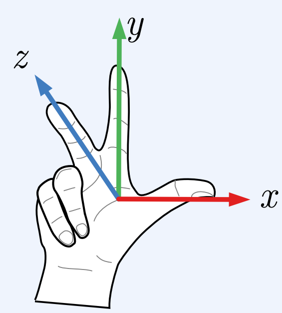

Figure 2.4.3 Right-handed coordinate system.

These are the labels for the three axes and your fingers point in their positive directions. If it is more convenient, you may name your thumb y

or z, as long as you name the other two fingers in the same sequence y-z-x or z-x-y.

2.4.2 Direction Cosine Angles

The direction of a vector in two-dimensional systems could be expressed clearly with a single angle measured from a reference axis, but adding an additional dimension means that one angle is no longer enough.

One way to define the direction of a three-dimensional vector is by using **direction cosine angles**, also commonly known as **coordinate direction angles**. The direction cosine angles are the angles between the positive x, y, and z axes to a given vector and are traditionally named θ_x, θ_y, and θ_z. Three-dimensional vectors, components, and angles are often difficult to visualize because they do not commonly lie in the Cartesian planes.

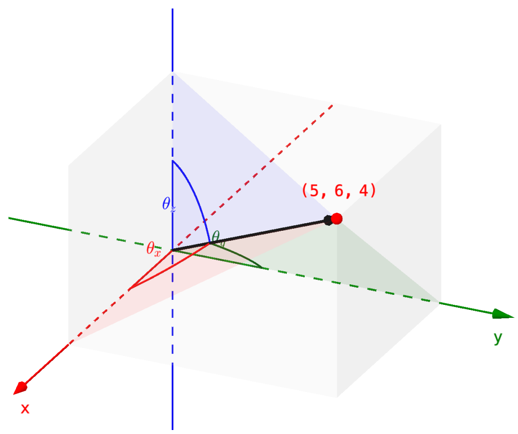

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Figure 2.4.4 Direction Cosine Angles

We can relate the components of a vector to its direction cosine angles using the following equations.

$$\cos\theta_x = \frac{A_x}{|A|} \qquad \cos\theta_y = \frac{A_y}{|A|} \qquad \cos\theta_z = \frac{A_z}{|A|} \tag{2.4.1}$$

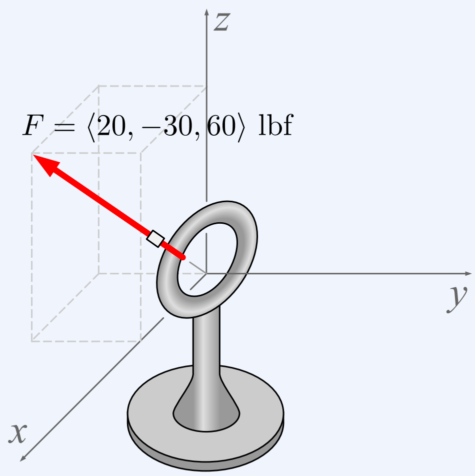
Note the component in the numerator of each direction cosine equation is positive or negative as defined by the coordinate system, and the vector magnitude in the denominator is always positive. From these equations, we can conclude that:

- Direction cosines are signed value between -1 and 1.
- Direction cosine angles must always be between 0° and 180° or

$$0° \leq \theta_n \leq 180°.$$

- Any direction cosine angle greater than 90° indicates a negative component along that respective axis. Spatially this is because all direction cosine angles are measured from the positive side of each axis. Mathematically this is because the cosine of any angle between 90 and 180 degrees is numerically negative.

Example 2.4.5 Direction Cosine Angles.



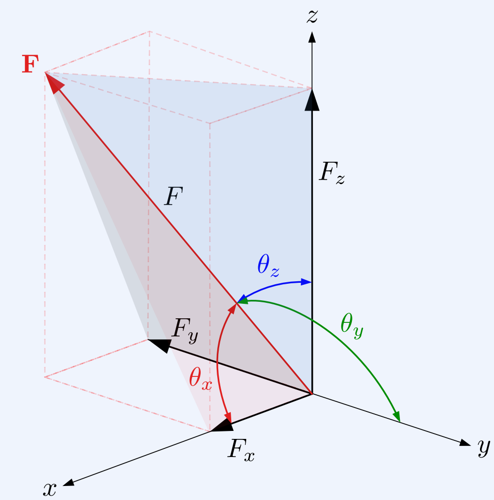
A rope pulls on an anchor ring centered at the origin with force $\mathbf{F} = \langle 20, -30, 60 \rangle$ lbf.
Find the magnitude of $\mathbf{F}$ and the direction cosine angles, θ_x, θ_y, and θ_z components.

Solution. Since the three components of F are perpendicular, we can apply the Pythagorean Theorem to find the magnitude of F.

$$\begin{aligned} F = |\mathbf{F}| &= \sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2} \\ &= \sqrt{20^2 + (-30)^2 + 60^2} \text{ lbf} \\ &= 70 \text{ lbf} \end{aligned}$$

Direction cosine angles are equal to the inverse cosine of each Cartesian force component divided by the force magnitude.

$$\begin{aligned} \theta_x &= \cos^{-1}\left(\frac{F_x}{|\mathbf{F}|}\right) = \cos^{-1}\left(\frac{20}{70}\right) = 73.4^\circ \\ \theta_y &= \cos^{-1}\left(\frac{F_y}{|\mathbf{F}|}\right) = \cos^{-1}\left(\frac{-30}{70}\right) = 115.38^\circ \\ \theta_z &= \cos^{-1}\left(\frac{F_z}{|\mathbf{F}|}\right) = \cos^{-1}\left(\frac{60}{70}\right) = 31.0^\circ \end{aligned}$$



Since the direction cosine angles are measured from the positive x, y, and z axes, the negative component of F_y means that $\theta_y > 90°$, while both θ_x and θ_z are less than $90°$ as their components are positive.

2.4.3 Spherical Coordinates

In spherical coordinates, points are specified with these three coordinates

- r, the **radial distance** from the origin to the tip of the vector,
- θ, the **azimuthal angle**, measured counter-clockwise from the positive x axis to the projection of the vector onto the xy plane, and
- ϕ, the **polar angle** from the z axis to the vector.

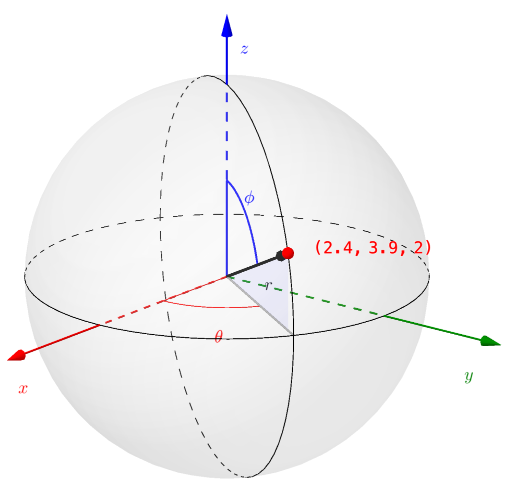


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Figure 2.4.6 Spherical Coordinate System

Question 2.4.7

What are the differences between polar coordinates and terrestrial latitude/longitude locations?

Answer. In terrestrial measurements

- Coordinate r is not needed since all points are on the surface of the globe.
- Longitude is measured $0°$ to $180°$ East or West of the prime meridian, rather than $0°$ to $360°$ counter-clockwise from the x axis.
- Latitude is measured $0°$ to $90°$ North or South of the equator, whereas polar angle ϕ is $0°$ to $180°$ measured from the "North Pole".

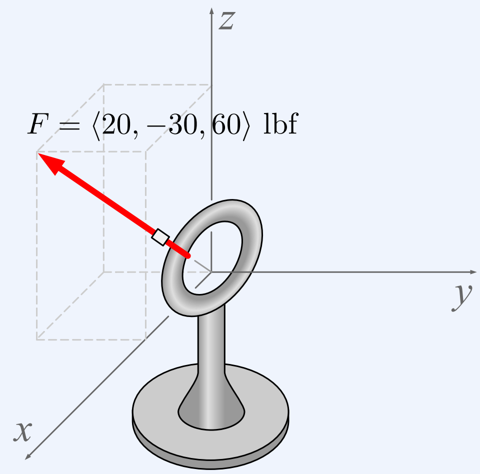
When vectors are specified using cylindrical coordinates the magnitude of the vector is used instead of distance r from the origin to the point.

When the two given spherical angles are defined in the manner shown here, the rectangular components of the vector $\mathbf{A} = (A\ ; \theta\ ; \phi)$ are found thus:

$$A' = A \sin\phi \tag{2.4.2}$$
$$A_z = A \cos\phi \tag{2.4.3}$$
$$A_x = A' \cos\theta = A \sin\phi \cos\theta \tag{2.4.4}$$
$$A_y = A' \sin\theta = A \sin\phi \sin\theta \tag{2.4.5}$$

Reflect on the equations above. Can you think through the process of how they were derived? The generalized steps are as follows. First, draw an accurate sketch of the given information and define the right triangles related to both θ and ϕ. Then use trig identities on the right triangle involving the vector, the z axis and angle ϕ to find A_z, and A', the projection of $\mathbf{A}$ onto the xy plane. Finally, use trig identities on the right triangle involving vector $\mathbf{A}'$ and θ to find the remaining components of $\mathbf{A}$.

Example 2.4.8 Spherical Coordinates.

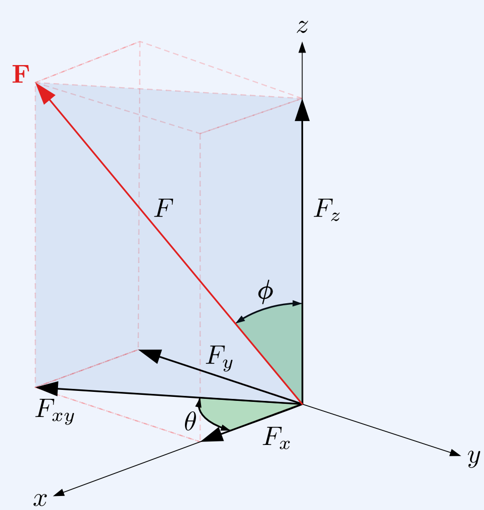


A rope pulls on an anchor ring centered at the origin with force $\mathbf{F} = \langle 20, -30, 60\rangle$ lbf.
Find the *spherical* coordinates of $\mathbf{F}$.

Solution. To represent $\mathbf{F}$ in spherical coordinates, we must find the radial distance r, the azimuthal angle θ, and the polar angle ϕ.
Coordinate r is simply the magnitude of force $\mathbf{F}$. Since the three components of $\mathbf{F}$ are perpendicular, we can apply the Pythagorean Theorem to find it.

$$\begin{aligned} F = |\mathbf{F}| = r &= \sqrt{F_x^2 + F_y^2 + F_z^2} \\ &= \sqrt{20^2 + (-30)^2 + 60^2} \text{ lbf} \\ &= 70 \text{ lbf} \end{aligned}$$

Azimuthal angle θ measures the angle between the x axis and the projection of $\mathbf{F}$ onto the xy plane, F_{zy}.
Using a right triangle with sides F_x, F_y, and F_{xy}, we can find θ using the inverse tangent of the ratio of the opposite to adjacent sides.

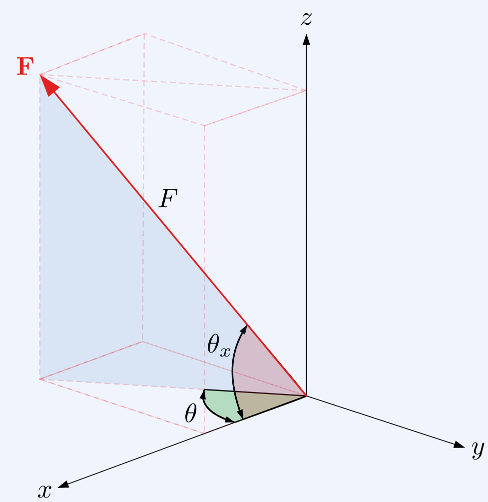


$$\theta = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\left(\frac{-30}{20}\right) = -56.31°$$

This angle is negative because it is measured *clockwise* from the positive x axis, opposite the standard CCW direction.
The polar angle ϕ is measured down from the $+z$ axis to the vector $\mathbf{F}$. We can find it using a right triangle with sides F, F_z, and F_{xy}. Note that ϕ is the same as the direction cosine angle θ_z.

$$\phi = \theta_z = \cos^{-1}\frac{F_z}{|\mathbf{F}|} = \cos^{-1}\left(\frac{60}{70}\right) = 31.0°$$

Also notice that the azimuthal angle θ is smaller than the direction cosine angle θ_x, since θ is in the xy plane, but θ_x is a 3D angle from the x axis to the vector $\mathbf{F}$.



2.4.4 Cylindrical Coordinates

The cylindrical coordinate system is seldom used in statics, however, it is useful in certain geometries. Cylindrical coordinates extend two-dimensional polar coordinates by adding a z coordinate indicating the distance above or below the xy plane.

Points are specified with these three cylindrical coordinates.

- r, the radius of the cylinder. This is the distance from the origin to the projection of the tip of the vector onto the xy plane,
- θ, the azimuthal angle, measured counter-clockwise from the positive x axis to the projection of the vector onto the xy plane
- z, the vertical height of the vector tip.

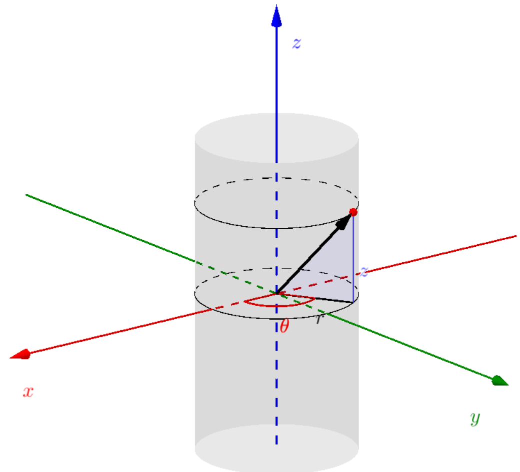


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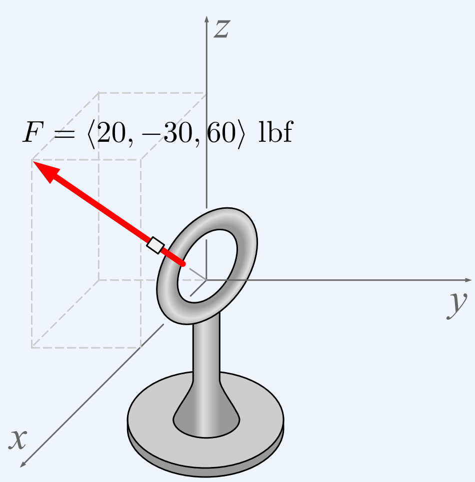
Figure 2.4.9 Cylindrical Coordinate System

Unfortunately, not all problems give the angles θ and ϕ as defined here; so you will need to find them from the given angles in other situations.

You can use the interactive diagram in this section to practice visualizing and finding the components of a vector in all of these coordinate systems. You should be able to find the x, y, and z coordinates given direction angles, spherical or cylindrical coordinates, and vice-versa.

Example 2.4.10 Cylindrical Coordinates.

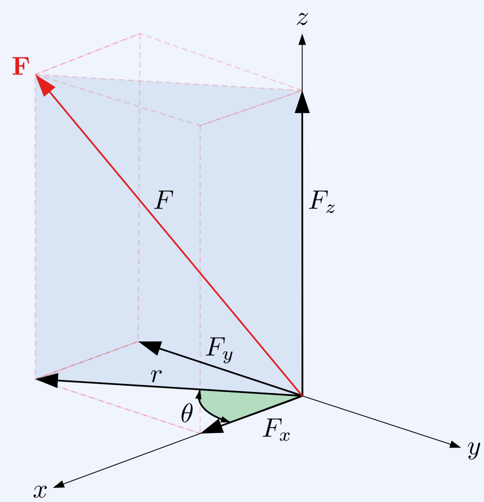


A rope pulls on an anchor ring centered at the origin with force $\mathbf{F} = \langle 20, -30, 60 \rangle$ lbf.
Find the *cylindrical* coordinates of $\mathbf{F}$.

Solution. To represent $\mathbf{F}$ in cylindrical coordinates, we must find the radial distance, r, the azimuthal angle, θ, and the axial coordinate, z.
In cylindrical coordinates, r is the radius of the cylinder rather than the radius of the enclosing sphere. r is the projection of $\mathbf{F}$ onto the xy plane, F_{xy}, and can be found by applying the Pythagorean Theorem to the x and y components of $\mathbf{F}$.

$$
\begin{aligned}
r = F_{xy} &= \sqrt{F_x^2 + F_y^2} \\
&= \sqrt{20^2 + (-30)^2} \text{ lbf} \\
&= 36.06 \text{ lbf}
\end{aligned}
$$

The azimuthal angle θ is the same in both cylindrical and spherical coordinates. It measures the angle between the x axis and the projection of **F** onto the xy plane. θ can be found using a right triangle in the xy plane with sides F_x and F_y.



$$\theta = \tan^{-1} \frac{F_y}{F_x} = \tan^{-1} \left(\frac{-30}{20} \right) = -56.31°$$

Finally, the z component is the vertical component of the force, F_z, which was given.

$$F_z = 60.0 \text{ lbf}$$

2.5 Unit Vectors

Key Questions

- Why are unit vectors useful?
- What are the unit vectors along the Cartesian x, y, and z axes?
- How do you find the force vector components of known force magnitude along a geometric line?
- How can you find unit vector components from direction cosine an-

gles?

A unit vector is a vector with a magnitude of one and no units. As such, a unit vector represents a pure direction. By convention, a unit vector is indicated by a *hat* over a vector symbol. This may sound like a new concept, but it's a simple one, directly related to the unit circle, the Pythagorean Theorem, and the definitions of sine and cosine.

2.5.1 Cartesian Unit Vectors

A unit vector can point in any direction, but because they occur so frequently the unit vectors in each of the three Cartesian coordinate directions are given their own symbols, which are:

- **i**, for the unit vector pointing in the x direction,
- **j**, for the unit vector pointing in the y direction, and
- **k**, for the unit vector pointing in the z direction..

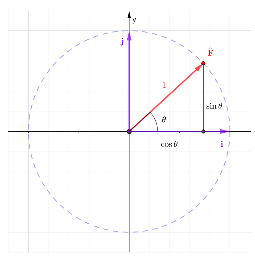


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Figure 2.5.1 Unit Vector Interactive

Applying the Pythagorean Theorem to the triangle gives the equation for a unit circle

$$\cos^2\theta + \sin^2\theta = 1^2$$

No matter what angle a unit vector makes with the x axis, $\cos\theta$ and $\sin\theta$ are its scalar components. This relation assumes that the angle θ is measured from the x axis, if it is measured from the y axis the sine and cosine functions reverse, with $\sin\theta$ defining the horizontal component and the $\cos\ \theta$ defining the vertical component.

The x and y components of a point on the unit circle are also the scalar components of $\hat{\mathbf{F}}$, so

$$\left.\begin{aligned} F_x &= \cos\theta \\ F_y &= \sin\theta \end{aligned}\right\} \implies \hat{\mathbf{F}} = \langle \cos\theta, \sin\theta \rangle.$$

2.5.2 Relation between Vectors and Unit Vectors

When a purely-directional unit vector is multiplied by a scalar value it is *scaled* by that amount. For example, when a unit vector pointing to the right is multiplied by 100 N the result is a 100 N vector pointing to the right. When a unit vector pointing up is multiplied by −50 N, the negative magnitude flips the direction of the unit vector and the result is a 50 N vector pointing down.

In general,

$$\mathbf{F} = |\mathbf{F}|\ \hat{\mathbf{F}}, \tag{2.5.1}$$

where $|\mathbf{F}|$ is the magnitude of vector $\mathbf{F}$, and $\hat{\mathbf{F}}$ is the unit vector pointing in the direction of $\mathbf{F}$.

Solving equation (2.5.1) for $\hat{\mathbf{F}}$ gives the approach to find the unit vector of known vector $\mathbf{F}$.

The process is straightforward— divide the vector by its magnitude.

$$\hat{\mathbf{F}} = \frac{\mathbf{F}}{|\mathbf{F}|} \tag{2.5.2}$$

To emphasize that unit vectors are pure direction, recall that vectors consist of both a magnitude and direction, so when we divide a vector by its own magnitude we are just left with direction.

$$\text{unit vector} = \frac{\mathbf{F}}{|\mathbf{F}|} = \frac{[\text{vector}]}{[\text{magnitude}]} = \frac{\cancel{[\text{magnitude}]}\cdot[\text{direction}]}{\cancel{[\text{magnitude}]}} = [\text{direction}]$$

This interactive shows vector $\mathbf{F}$, its associated unit vector $\hat{\mathbf{F}}$, and expressions for $\mathbf{F}$ in terms of its unit vector $\hat{\mathbf{F}}$.

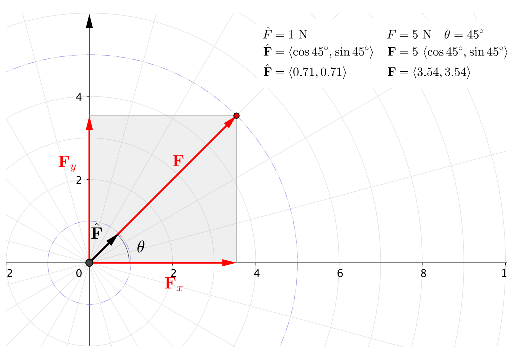

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Figure 2.5.2 Unit Vectors

Example 2.5.3 Find unit vector of a force.

Find the unit vector corresponding to a 100 N force at 60° above the positive x-axis.

Solution. In polar coordinates, the unit vector is a vector of magnitude 1, pointing in the same direction as the force, so, by inspection

$$\mathbf{F} = (100\ \text{N}\,; 60°)$$

$$\hat{\mathbf{F}} = (1\,;60°)$$

In rectangular coordinates, first express $\mathbf{F}$ in terms of its x and y components.

$$\left.\begin{aligned} F_x &= F\cos 60° \\ F_y &= F\sin 60° \end{aligned}\right\} \implies \mathbf{F} = \langle F\cos 60°, F\sin 60°\rangle.$$

$$\hat{\mathbf{F}} = \frac{\mathbf{F}}{F} = \frac{\langle 100\text{ N}\cos 60°, 100\text{ N}\sin 60°\rangle}{100\text{ N}} = \langle \cos 60°, \sin 60°\rangle$$

2.5.3 Force Vectors from Position Vectors

Unit vectors are generally the best approach when working with forces and distances in three dimensions.

For example, when the locations of two points on the line of action of a force are known, the unit vector of the line of action can be found and used to determine the components of the force acting along that line. This can be accomplished as follows, where A and B are points on the line of action.

1. Use the problem geometry to find $\mathbf{AB}$, the displacement vector from point A to point B.

   You can either subtract the coordinates of the starting point A from the coordinates of the destination point B,

   $$\begin{aligned} A &= (A_x, A_y, A_z) \\ B &= (B_x, B_y, B_z) \\ \mathbf{AB} &= (B_x - A_x)\,\mathbf{i} + (B_y - A_y)\,\mathbf{j} + (B_z - A_z)\,\mathbf{k}, \text{ or} \end{aligned}$$

   or, write the displacements directly by noting the change in the x, y, and z coordinates when moving from A to B.

   $$\begin{aligned} AB_x &= \Delta x = B_x - A_x \\ AB_y &= \Delta y = AB_y = B_y - A_y \\ AB_z &= \Delta z = B_z - A_x \\ \mathbf{AB} &= AB_x\ \mathbf{i} + AB_y\ \mathbf{j} + AB_z\ \mathbf{k} \end{aligned}$$

   The result is the same with either method.

2. Find the distance between point A and point B using the Pythagorean Theorem. This distance is also the magnitude of $\mathbf{AB}$ or $|\mathbf{AB}|$.

   $$|\mathbf{AB}| = \sqrt{(AB_x)^2 + (AB_y)^2 + (AB_z)^2}$$

3. Find $\widehat{\mathbf{AB}}$, the unit vector from A to B, by dividing vector $\mathbf{AB}$ by its magnitude. This is a unitless vector with a magnitude of 1 which points from A to B.

$$\widehat{\mathbf{AB}} = \left\langle \frac{AB_x}{|\mathbf{AB}|}, \frac{AB_y}{|\mathbf{AB}|}, \frac{AB_z}{|\mathbf{AB}|} \right\rangle$$

4. Finally, multiply the magnitude of the force by the unit vector $\widehat{\mathbf{AB}}$ to get force $\mathbf{F}_{AB}$.

$$\begin{aligned}\mathbf{F}_{AB} &= F_{AB}\,\widehat{\mathbf{AB}}\\ &= F_{AB}\left\langle \frac{AB_x}{|\mathbf{AB}|}, \frac{AB_y}{|\mathbf{AB}|}, \frac{AB_z}{|\mathbf{AB}|} \right\rangle\end{aligned}$$

The interactive below can be used to visualize the displacement vector and its unit vector, and practice this procedure.

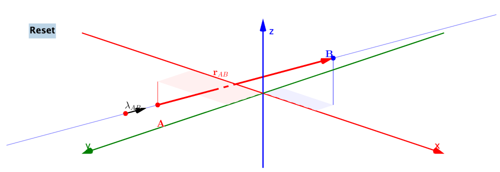

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Figure 2.5.4 Unit Vectors in Space

Example 2.5.5 Component in a Specified Direction.

Determine the components of a 5 kN force $\mathbf{F}$ acting at point A, in the direction of a line from A to B. Given: $A = (2, 3, -2.1)$ m and $B = (-2.5, 1.5, 2.2)$ m

We will take the solution one step at a time.

(a) Draw a good diagram.

Hint. The interactive in Figure 2.5.4 may be useful for this problem.

(b) Find the displacement vector from A to B.

Answer.

$$\mathbf{AB} = \langle -4.5, -1.5, 4.3 \rangle \text{ m}$$

Solution.

$$\mathbf{AB} = (B_x - A_x)\,\mathbf{i} + (B_y - A_y)\,\mathbf{j} + (B_z - A_z)\,\mathbf{k}$$

$$
\begin{aligned}
&= [(-2.5-2)\,\mathbf{i} + (1.5-3)\,\mathbf{j} + (2.2-(-2.1))\,\mathbf{k}] \;\; \text{m} \\
&= (-4.5\mathbf{i} - 1.5\mathbf{j} + 4.3\mathbf{k}) \;\; \text{m} \\
&= \langle -4.5, -1.5, 4.3 \rangle \;\text{m}
\end{aligned}
$$

(c) Find the magnitude of the displacement vector.

Answer.

$$|\mathbf{AB}| = 6.402 \text{ m}$$

Solution.

$$
\begin{aligned}
|\mathbf{AB}| &= \sqrt{(\Delta_x)^2 + (\Delta_y)^2 + (\Delta_z)^2} \\
&= \sqrt{(-4.5)^2 + (-1.5)^2 + 4.3^2 \text{ m}^2} \\
&= \sqrt{40.99 \text{ m}^2} \\
&= 6.402 \text{ m}
\end{aligned}
$$

(d) Find the unit vector pointing from A to B.

Answer.

$$\widehat{\mathbf{AB}} = \langle -0.7, -0.23, 0.67 \rangle$$

Solution.

$$
\begin{aligned}
\widehat{\mathbf{AB}} &= \left\langle \frac{\Delta_x}{|\mathbf{AB}|}, \frac{\Delta_y}{|\mathbf{AB}|}, \frac{\Delta_z}{|\mathbf{AB}|} \right\rangle \\
&= \left\langle \frac{-4.5}{6.402}, \frac{-1.5}{6.402}, \frac{4.3}{6.402} \right\rangle \\
\widehat{\mathbf{AB}} &= \langle -0.7, -0.23, 0.67 \rangle
\end{aligned}
$$

(e) Find the force vector.

Answer.

$$\mathbf{F}_{AB} = \langle -3.51, -1.17, 3.36 \rangle \;\; \text{kN}$$

Solution.

$$
\begin{aligned}
\mathbf{F}_{AB} &= F_{AB}\, \widehat{\mathbf{AB}} \\
&= 5 \text{ kN}\, \langle -0.7, -0.23, 0.67 \rangle \\
&= \langle -3.51, -1.17, 3.36 \rangle \;\; \text{kN}
\end{aligned}
$$

Given the properties of unit vectors, there are some conceptual checks you can make after computing unit vector components which can prevent subsequent errors.

- The signs of unit vector components need to match the signs of the original position vector. A unit vector has the same line of action and sense as the position vector but is scaled down to one unit in magnitude.
- Components of a unit vector must be between -1 and 1. If the magnitude of a unit vector is one, then it is impossible for it to have rectangular components larger than one.

2.5.4 Unit Vectors and Direction Cosines

If you look closely at the right side of equation (2.4.1), you will see that each equation consists of a component divided by the total vector magnitude. These are the same equations just used to find unit vectors. Thus, the cosine of each direction cosine angle collectively also computes the components of the unit vector; hence we can write an equation for $\hat{\mathbf{A}}$,i.e., the unit vector along $\mathbf{A}$.

$$\hat{\mathbf{A}} = \cos\theta_x\ \mathbf{i} + \cos\theta_y\ \mathbf{j} + \cos\theta_z\ \mathbf{k}$$

Combining the Pythagorean Theorem with our knowledge of unit vectors and direction cosine angles gives this result: if you know two of the three direction cosine angles you can manipulate the following equation to find the third.

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1 \tag{2.5.3}$$

2.6 Vector Addition

Key Questions

- How do you set up vectors for graphical addition using the Triangle Rule?
- Does it matter which vector you start with when using the Triangle Rule?
- Why can you separate a two-dimensional vector equation into two independent equations to solve for up to two unknowns?
- If you and another student define vectors using different direction coordinate systems, will you end up with the same resultant vector?
- What is the preferred technique to add vectors in three-dimensional

systems?

In this section we will look at several different methods of vector addition. Vectors being added together are called the **components**, and the sum of the components is called the **resultant vector**.

These different methods are tools for your statics toolbox. They give you multiple different ways to think about vector addition and to approach a problem. Your goal should be to learn to use them all and to identify which approach will be the easiest to use in a given situation.

2.6.1 Triangle Rule of Vector Addition

All methods of vector addition are ultimately based on the tip-to-tail method discussed in a one-dimensional context in Subsection 2.2.1. There are two ways to draw or visualize adding vectors in two or three dimensions, the **Triangle Rule** and **Parallelogram Rule**. Both are equivalent.

- *Triangle Rule.*

  Place the tail of one vector at the tip of the other vector, then draw the resultant from the first vector's tail to the final vector's tip.

- *Parallelogram Rule.*

  Place both vectors' tails at the origin, then complete a parallelogram with lines parallel to each vector through the tip of the other. The resultant is equal to the diagonal from the tails to the opposite corner.

The interactive below shows two forces **A** and **B** pulling on a particle at the origin, and the appropriate diagram for the triangle or parallelogram rule. Both approaches produce the same resultant force **R** as expected.

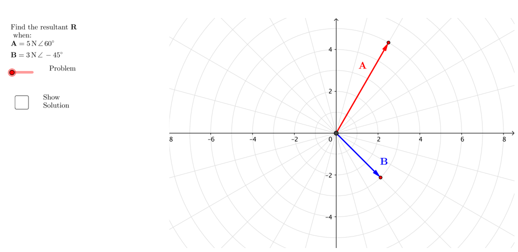

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Figure 2.6.1 Vector Addition Methods

2.6.2 Graphical Vector Addition

Graphical vector addition involves drawing a scaled diagram using either the parallelogram or triangle rule, and then measuring the magnitudes and directions from the diagram. Graphical solutions work well enough for two-dimensional problems where all the vectors live in the same plane and can be drawn on a sheet of paper, but are not very useful for three-dimensional problems unless you use technology.

If you carefully draw the triangle accurately to scale and use a protractor and ruler you can measure the magnitude and direction of the resultant. However, your answer will only be as precise as your diagram and your ability to read your tools. If you use technology such as GeoGebra or a CAD program to make the diagram, your answer will be precise. The interactive in Figure 2.6.1 may be useful for this.

Even though the graphical approach has limitations, it is worth your attention because it provides a good way to visualize the effects of multiple forces, to quickly estimate ballpark answers, and to visualize the diagrams you need to use alternate methods to follow.

2.6.3 Trigonometric Vector Addition

You *can* get a precise answer from the triangle or parallelogram rule by

1. drawing a quick diagram using either rule,
2. identifying three known sides or angles,
3. using trigonometry to solve for the unknown sides and angles.

The trigonometric tools you will need are found in Appendix **??**.

Using triangle-based geometry to solve vector problems is a quick and powerful tool, but includes the following limitations:

- There are only three sides in a triangle; thus vectors can only be added two at a time. If you need to add three or more vectors using this method, you must add the first two, then add the third to that sum and so on.
- If you fail to draw the correct vector triangle or identify the known sides and angles, you will not find the correct answer.
- The trigonometric functions produce scalar values. You can use them to find the magnitudes the angles between vectors, but the results are not, by themselves, vectors.

When you need to find the resultant of more than two vectors, it is generally best to use the algebraic methods described below.

2.6.4 Orthogonal Components

Any arbitrary two-dimensional vector $\mathbf{F}$ can be broken into two component vectors which are the sides of a parallelogram having $\mathbf{F}$ as its diagonal. The process of finding components of a vector in particular directions is called **vector resolution**. While a vector can be resolved into components in any two directions, it's generally most useful to resolve them into **rectangular** or **orthogonal** components, where the parallelogram is a rectangle and the components are perpendicular.

There are an infinite number of possible rectangles to choose from, so each vector has an infinite number of sets of orthogonal components. However, the most important set occurs when the sides of the rectangle are parallel to the x and y axes. These orthogonal components are given x and y subscripts indicate that they're aligned with the coordinate axes. For vector $\mathbf{F}$,

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y = F_x\ \mathbf{i} + F_y\ \mathbf{j}, \tag{2.6.1}$$

where F_x and F_y are the scalar components of $\mathbf{F}$. The advantage of this choice of components is that vector calculations can be replaced with ordinary algebric calculation on scalar values for each orthogonal direction.

Alternately, you may rotate the coordinate system to any other convenient angle, and find the components in the directions of the rotated coordinate axes x' and y'. In either case, the vector is the sum of the rectangular components

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y = \mathbf{F}_{x'} + \mathbf{F}_{y'}. \tag{2.6.2}$$

The interactive below can help you visualize the relationship between a vector and its components in both the x-y and x'-y' directions.

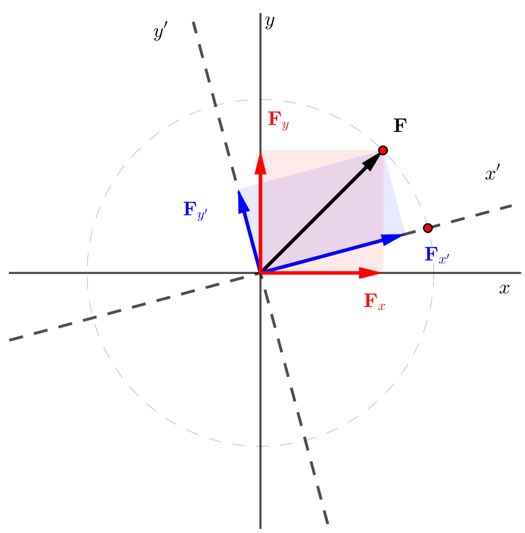

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Figure 2.6.2 Orthogonal Components

2.6.5 Algebraic Addition of Components

While the parallelogram rule and the graphical and trigonometric methods are useful tools for visualizing and finding the sum of two vectors, they are not particularly suited for adding *more* than two vectors or working in three dimensions.

Consider vector **R** which is the sum of several vectors **A**, **B**, **C** and perhaps more. Vectors **A**, **B** and **C** are the components of **R**, and the **R** is the resultant of **A**,**B** and **C**.

It is easy enough to say that $\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C}$, but how can we calculate **R**? You could draw the vectors arranged tip-to-tail and then use the triangle rule to add the first two components, then use it again to add the third component to that sum, and so forth until all the components have been added. The final sum is the resultant, **R**. The process gets progressively more tedious the more components there are to sum.

This section introduces an alternate method to add multiple vectors which is straightforward, efficient and robust. This is called *algebraic method*, because the vector addition is replaced with a process of *algebraic* addition of *scalar* components. The algebraic technique works equally well for two and three-dimensional vectors, and for summing any number of vectors.

To find the sum of multiple vectors using the algebraic method:

1. Find the scalar components of each component vector in the x and y directions using the P to R procedure described in Subsection 2.3.3.

2. Algebraically sum the scalar components in each coordinate direction. The scalar components will be positive if they point right or up, negative if they point left or down. These sums are the scalar components of the resultant.

3. Resolve the resultant's components to find the magnitude and direction of the resultant vector using the R to P procedure described in Subsection 2.3.3.

The resultant $\mathbf{F}_R$ is the simply the algebraic sum of the components in each coordinate direction.

$$\mathbf{F}_R = \Sigma F_x \; \mathbf{i} + \Sigma F_y \; \mathbf{j} + \Sigma F_z \; \mathbf{k}$$

or in bracket notation

$$\mathbf{F}_R = \langle \Sigma F_x, \Sigma F_y, \Sigma F_z \rangle \,. \tag{2.6.3}$$

This process is illustrated in the following interactive diagram and in the next example.

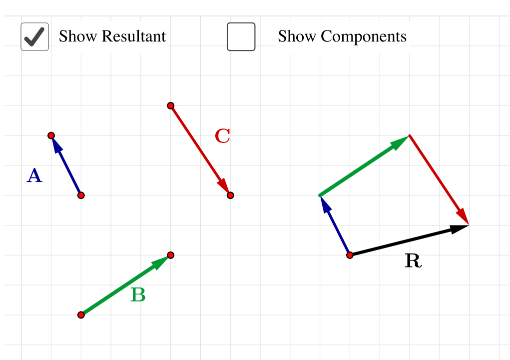

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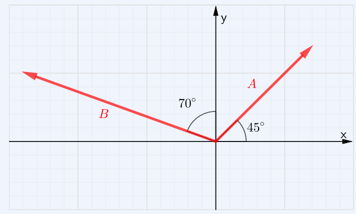
Figure 2.6.3 Vector addition by summing rectangular components.

Example 2.6.4 Vector Addition.

Vector $\mathbf{A} = 200$ N$\angle$ 45° counter-clockwise from the x axis, and vector $\mathbf{B} = 300$ N $\angle 70°$ counter-clockwise from the y axis.
Find the resultant $\mathbf{R} = \mathbf{A} + \mathbf{B}$ by addition of scalar components.

Solution.

Use the given information to draw a sketch of the situation. By imagining or sketching the parallelogram rule, it should be apparent that the resultant vector points up and to the left.



$$A_x = 200 \text{ N} \cos 45° = 141.4 \text{ N} \qquad B_x = -300 \text{ N} \sin 70° = -281.9 \text{ N}$$
$$A_y = 200 \text{ N} \sin 45° = 141.4 \text{ N} \qquad B_y = 300 \text{ N} \cos 70° = 102.6 \text{ N}$$

$$\begin{aligned} R_x &= A_x + B_x & R_y &= A_y + B_y \\ &= 141.4 \text{ N} - 281.9 \text{ N} & &= 141.4 \text{ N} + 102.6 \text{ N} \\ &= -140.5 \text{ N} & &= 244.0 \text{ N} \end{aligned}$$

$$R = \sqrt{R_x^2 + R_y^2} = 281.6 \text{ N} \qquad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = -60.1°$$

This answer indicates that the resultant points down and to the left, which is odd because the parallelogram rule shows that the resultant should point up and to the left.
This occurs because the calculator always returns angles in the first or fourth quadrant for $\tan^{-1}$. To get the actual direction of the resultant, add 180° to the calculator result.

$$\theta = -60.1° + 180° = 119.9°$$

The final answer for the magnitude and direction of the resultant is

$$\mathbf{R} = 281.6 \text{ N}\angle 119.9°$$

measured counter-clockwise from the x axis.

The process for adding three-dimensional vectors is exactly the same, except that the z component is included as well. This interactive allows you to input the three-dimensional vector components of forces $\mathbf{A}$ and $\mathbf{B}$ and view the resultant force $\mathbf{R}$ which is the sum of $\mathbf{A}$ and $\mathbf{B}$.

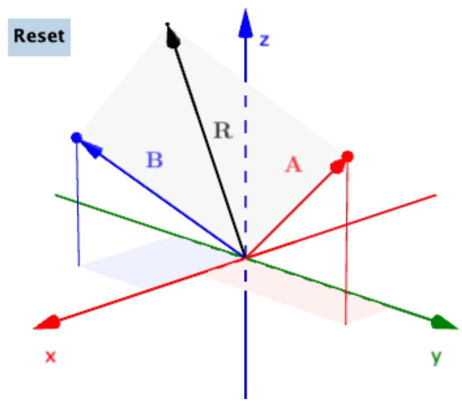


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Figure 2.6.5 Vector Addition in Three Dimensions

2.6.6 Vector Subtraction

Like one-dimensional vector subtraction, the easiest way to handle two-dimensional vector subtraction is by taking the negative of a vector followed by vector addition. Multiplying a vector by -1 preserves its magnitude but flips its direction, which has the effect of changing the sign of the scalar components.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

After negating the second vector you can choose any technique you prefer for vector addition.

2.7 Dot Products

Key Questions

- What are dot products used for?
- What does it mean when the dot product of two vectors is zero?
- How do you use a dot product to find the angle between two vectors?
- What does it mean when the scalar component of the projection $\|\operatorname{proj}_{\mathbf{A}} \mathbf{B}\|$ is negative?

Unlike ordinary algebra where there is only one way to multiply numbers, there are two distinct vector multiplication operations: **dot product** and the **cross product**. Alternately, the first is referred to as the **scalar product** because its result is a scalar, and the second as the **vector product** because its result is a vector. The dot product and its applications will be discussed in this section and the cross product in the next.

2.7.1 Calculation of the Dot product

For two vectors $\mathbf{A} = \langle A_x, A_y, A_z\rangle$ and $\mathbf{B} = \langle B_x, B_y, B_z\rangle$, the dot product multiplication is computed by summing the products of the components.

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{2.7.1}$$

It can be shown that an alternate, equivalent method to compute the dot product is

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = A\ B\ \cos\theta \tag{2.7.2}$$

where θ in the equation is the angle between the two vectors and $|\mathbf{A}|$ and $|\mathbf{B}|$ are the magnitudes of $\mathbf{A}$ and $\mathbf{B}$.

We can conclude from the second equation that the dot product of two perpendicular vectors is zero, because $\cos 90° = 0$, and that the dot product of two parallel vectors equals the product of their magnitudes.

When dotting unit vectors that have a magnitude of one, the dot products of a unit vector with itself is one and the dot product two perpendicular unit vectors is zero, so for $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ we have

$$\begin{array}{ccc} \mathbf{i}\cdot\mathbf{i} = 1 & \mathbf{j}\cdot\mathbf{i} = 0 & \mathbf{k}\cdot\mathbf{i} = 0 \\ \mathbf{i}\cdot\mathbf{j} = 0 & \mathbf{j}\cdot\mathbf{j} = 1 & \mathbf{k}\cdot\mathbf{j} = 0 \\ \mathbf{i}\cdot\mathbf{k} = 0 & \mathbf{j}\cdot\mathbf{k} = 0 & \mathbf{k}\cdot\mathbf{k} = 1 \end{array}$$

Dot products are commutative, associative and distributive:

1. **Commutative.** The order does not matter.

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \tag{2.7.3}$$

2. **Associative.** It does not matter whether you multiply a scalar value C by the final dot product, or either of the individual vectors, you will still get the same answer.

$$C\,(\mathbf{A}\cdot\mathbf{B}) = (C\ \mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(C\ \mathbf{B}) \tag{2.7.4}$$

3. **Distributive.** If you are dotting one vector $\mathbf{A}$ with the sum of two more $(\mathbf{B}+\mathbf{C})$, you can either add $\mathbf{B}+\mathbf{C}$ first, or dot $\mathbf{A}$ by both and add the final value.

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B}) + (\mathbf{A}\cdot\mathbf{C}) \tag{2.7.5}$$

Dot products can be used to compute the magnitude of a vector, determine the angle between two vectors, and find the rectangular component or projection of a vector in a specified direction. These applications will be discussed in the following sections.

2.7.2 Magnitude of a Vector

Dot products can be used to find vector magnitudes. When a vector is dotted with itself using (2.7.1), the result is the square of the magnitude of the vector. (Recall that $|\mathbf{A}|$ and A are alternate notations for the magnitude of vector $\mathbf{A}$.)

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2. \tag{2.7.6}$$

The proof is trivial. By the definition of the dot product (2.7.1) and the Pythagorean theorem:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{A} &= A_x A_x + A_y A_y \\ &= A_x^2 + A_y^2 = A^2 \\ &= |\mathbf{A}|^2 \end{aligned}$$

Taking the square root of each side gives the magnitude of $\mathbf{A}$

$$\sqrt{\mathbf{A} \cdot \mathbf{A}} = A = |\mathbf{A}|. \tag{2.7.7}$$

The result is similar for three-dimensional vectors.

Example 2.7.1 Find Vector Magnitude using the Dot Product.

Find the magnitude of vector $\mathbf{F}$ with components $F_x = 30$ N, $F_y = -40$ N and $F_z = 50$ N

Solution.

$$\mathbf{F} = \langle 30 \text{ N}, -40 \text{ N}, 50 \text{ N} \rangle$$

$$\begin{aligned} \mathbf{F} \cdot \mathbf{F} &= F_x^2 + F_y^2 + F_z^2 \\ &= (30 \text{ N})^2 + (-40 \text{ N})^2 + (50 \text{ N})^2 \\ &= 5000 \text{ N}^2 \end{aligned}$$

$$\begin{aligned} F &= |\mathbf{F}| = \sqrt{\mathbf{F} \cdot \mathbf{F}} \\ &= \sqrt{5000 \text{ N}^2} \\ &= 70.7 \text{ N} \end{aligned}$$

2.7.3 Angle between Two Vectors

A second application of the dot product is to find the angle between two vectors. Equation (2.7.2) provides the procedure.

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}||\mathbf{B}| \cos\theta \\ \cos\theta &= \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \end{aligned} \tag{2.7.8}$$

Example 2.7.2 Angle between Orthogonal Unit Vectors.

Find the angle between $\mathbf{i} = \langle 1, 0, 0\rangle$ and $\mathbf{j} = \langle 0, 1, 0\rangle$.

Solution.

$$\begin{aligned}\cos\theta &= \frac{\mathbf{i}\cdot\mathbf{j}}{|\mathbf{i}||\mathbf{j}|}\\ &= \frac{(1)(0) + (0)(1) + (0)(0)}{(1)(1)}\\ &= 0\\ \\ \theta &= \cos^{-1}(0)\\ &= 90^\circ\end{aligned}$$

This shows that $\mathbf{i}$ and $\mathbf{j}$ are perpendicular to each other.

Example 2.7.3 Angle between Two Vectors.

Find the angle between $\mathbf{F} = \langle 100\text{ N}, 200\text{ N}, -50\text{ N}\rangle$ and $\mathbf{G} = \langle -75\text{ N}, 150\text{ N}, -40\text{ N}\rangle$.

Solution.

$$\begin{aligned}\cos\theta &= \frac{\mathbf{F}\cdot\mathbf{G}}{|\mathbf{F}||\mathbf{G}|}\\ &= \frac{F_xG_x + F_yG_y + F_zG_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}\sqrt{G_x^2 + G_y^2 + G_z^2}}\\ &= \frac{(100)(-75) + (200)(150) + (-50)(-40)}{\sqrt{100^2 + 200^2 + (-50)^2}\sqrt{(-75)^2 + 150^2 + (-40)^2}}\\ &= \frac{24500}{(229.1)(172.4)}\\ &= 0.620\\ \\ \theta &= \cos^{-1}(0.620)\\ &= 51.7^\circ\end{aligned}$$

2.7.4 Vector Projection

The dot product is used to find the **projection** of one vector onto another. You can think of a projection of $\mathbf{B}$ on $\mathbf{A}$ as a vector the length of the shadow of $\mathbf{B}$ on the line of action of $\mathbf{A}$ when the sun is directly above $\mathbf{A}$. More precisely, the projection of $\mathbf{B}$ onto $\mathbf{A}$ produces the rectangular component of $\mathbf{B}$ in the direction *parallel* to $\mathbf{A}$. This is one side of a rectangle aligned with $\mathbf{A}$, having $\mathbf{B}$ as its diagonal.

This is illustrated in Figure 2.7.4, where $\mathbf{u}$ is the projection of $\mathbf{B}$ onto $\mathbf{A}$, or alternately $\mathbf{u}$ is the rectangular component of $\mathbf{B}$ in the direction of $\mathbf{A}$.

In this text we will use the symbols

- $\operatorname{proj}_{\mathbf{A}} \mathbf{B}$ to mean the **vector projection** of $\mathbf{B}$ on $\mathbf{A}$
- $|\operatorname{proj}_{\mathbf{A}} \mathbf{B}|$ to mean the *magnitude* of the vector projection, a positive or zero-valued scalar, and
- $\|\operatorname{proj}_{\mathbf{A}} \mathbf{B}\|$ to mean the **scalar projection**. This value represents the component of $\mathbf{B}$ in the $\mathbf{A}$ direction, and can have a positive, zero, or negative value.

As we have mentioned before, the magnitude of a vector is its length and is always positive or zero, while a scalar component is a signed value that can be positive or negative. When a scalar component is multiplied by a unit vector the result is a vector in that direction when the scalar component is positive, or 180° opposite when the scalar component is negative.

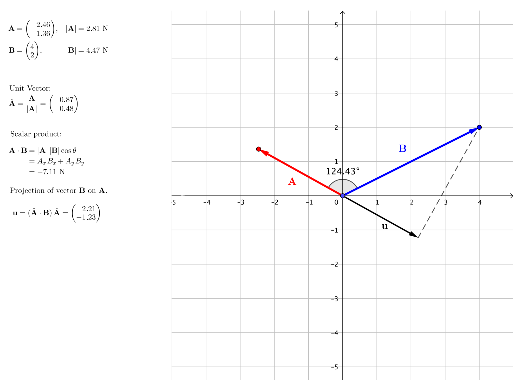

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Figure 2.7.4 Vector projection in two dimensions.

The interactive shows that the projection is the adjacent side of a right triangle with $\mathbf{B}$ as the hypotenuse. From the definition of the dot product (2.7.2) we find that

$$\mathbf{A} \cdot \mathbf{B} = A(B\ \cos\theta) = A\ \|\operatorname{proj}_{\mathbf{A}} \mathbf{B}\|, \tag{2.7.9}$$

where $B\ \cos\theta$ is the scalar component of the projection. So, the dot product of $\mathbf{A}$ and $\mathbf{B}$ gives us the projection of $\mathbf{B}$ onto $\mathbf{A}$ times the magnitude of $\mathbf{A}$. This value will be positive when $\theta < 90°$, negative when $\theta > 90°$, and zero when the vectors are perpendicular because of the properties of the cosine function.

So, to find the scalar value of the projection of $\mathbf{B}$ onto $\mathbf{A}$ we divide by the magnitude of $\mathbf{A}$.

$$\|\operatorname{proj}_{\mathbf{A}} \mathbf{B}\| = \frac{\mathbf{A} \cdot \mathbf{B}}{A} = \frac{\mathbf{A}}{A} \cdot \mathbf{B} = \hat{\mathbf{A}} \cdot \mathbf{B} \tag{2.7.10}$$

where $\hat{\mathbf{A}} = \dfrac{\mathbf{A}}{A}$ is the unit vector in the dirction of $\mathbf{A}$.

If you want the *vector* projection of **B** onto **A**, as opposed to the *scalar* projection we just found, multiply the *scalar* projection by the unit vector $\hat{\mathbf{A}}$.

$$\operatorname{proj}_{\mathbf{A}} \mathbf{B} = \|\operatorname{proj}_{\mathbf{A}} \mathbf{B}\|\hat{\mathbf{A}} = \left(\hat{\mathbf{A}} \cdot \mathbf{B}\right)\hat{\mathbf{A}} \tag{2.7.11}$$

Similarly, the vector projection of **A** onto **B** is

$$\operatorname{proj}_{\mathbf{B}} \mathbf{A} = \|\operatorname{proj}_{\mathbf{B}} \mathbf{A}\|\hat{\mathbf{B}} = \left(\mathbf{A} \cdot \hat{\mathbf{B}}\right)\hat{\mathbf{B}} \tag{2.7.12}$$

The spatial interpretation of the results the scalar projection $\|\operatorname{proj}_{\mathbf{A}} \mathbf{B}\|$ is

- *Positive values* mean that **A** and **B** are generally in the same direction.
- *Negative values* mean that **A** and **B** are generally in opposite directions.
- *Zero* means that **A** and **B** are perpendicular.
- *Magnitude smaller than* **B** This is the most common answer. This means that the vectors are neither parallel nor perpendicular.
- *Magnitude equal to* **B** means that the vectors point in the same direction, and all of **B** acts in the direction of **A**.
- *Magnitude larger than* **B** This answer is impossible. Check your algebra; you might have forgotten to divide by the magnitude of **A**.

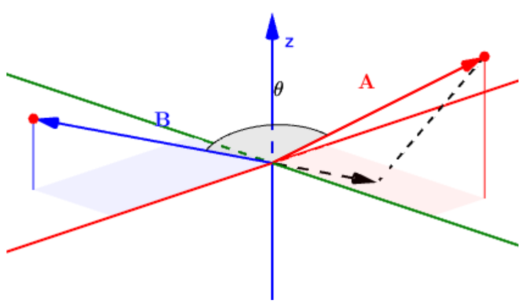

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Figure 2.7.5 Vector projections in three dimensions.

2.7.5 Perpendicular Components

The final application of dot products is to find the component of one vector perpendicular to another.

To find the component of **B** perpendicular to **A**, first find the vector projection of **B** on **A**, then subtract that from **B**. What remains is the perpendicular component.

$$\mathbf{B}_{\perp} = \mathbf{B} - \operatorname{proj}_{\mathbf{A}} \mathbf{B} \tag{2.7.13}$$

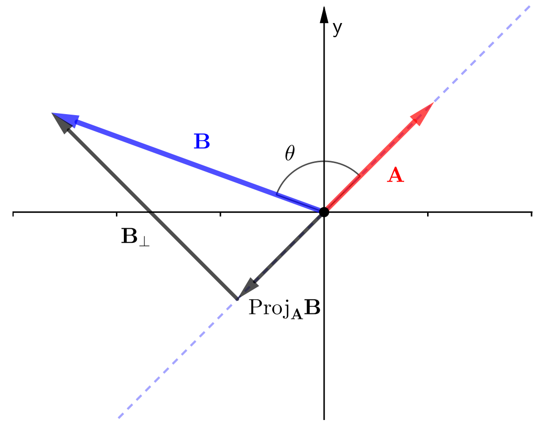
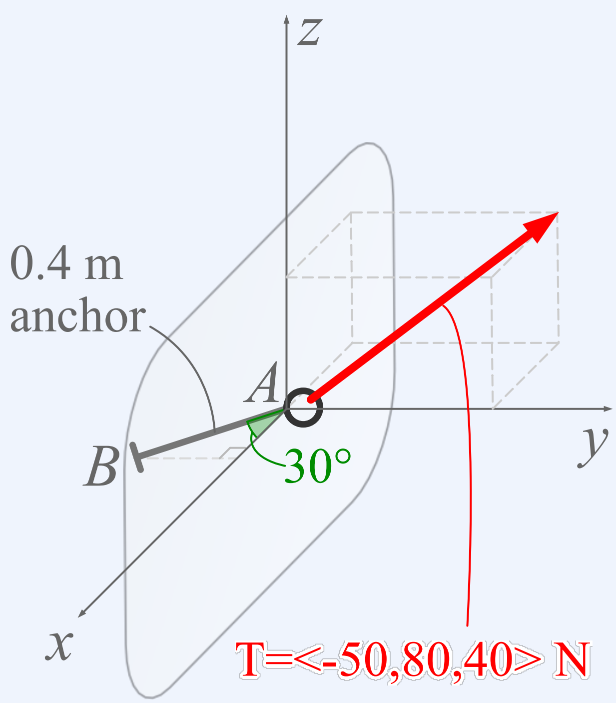


Figure 2.7.6 Perpendicular and parallel components of **B**.

Example 2.7.7 Dot Products.

A cable pulls with tension $\mathbf{T} = \langle -50, 80, 40 \rangle$ N on a 0.4 m long anchor AB embedded in a concrete wall. The anchor lies in the xy plane at an angle $\alpha = 30°$ from the x axis.



For the system above, compute the following:

(a) Find the dot product of the cable tension **T** and the anchor **AB**

Answer.

$$\mathbf{T} \cdot \mathbf{AB} = -33.32 \text{ N·m}$$

Solution. When you know the magnitudes and angle between two vectors, it is easiest to use the second dot product equation (2.7.2), but in this case it will be easier to find the components of **AB** and use (2.7.1).

$$\begin{aligned}\mathbf{T} &= \langle -50, 80, 40\rangle \text{ N}\\ \mathbf{AB} &= \langle 0.4\cos 30°, -0.4\sin 30°, 0\rangle \text{ m}\\ &= \langle 0.3464, -0.2, 0\rangle \text{ m}\end{aligned}$$

$$\begin{aligned}\mathbf{T}\cdot\mathbf{AB} &= T_x\,(AB_x) + T_y\,(AB_y) + T_z\,(AB_z)\\ &= (-50 \text{ N})\,(0.3464 \text{ m}) + (80 \text{ N})\,(-0.2 \text{ m}) + (40 \text{ N})\,(0 \text{ m})\\ &= -33.32 \text{ N}\cdot\text{m}\end{aligned}$$

The units of a dot product are the product of the units of the two vectors.

(b) Find the angle θ between the cable tension **T** and the anchor **AB**.

Answer.

$$\theta = 144.38°$$

Solution. With the dot product $\mathbf{T}\cdot\mathbf{AB}$ known from the previous step, we can use the (2.7.2) to find the angle between force **T** and anchor.

$$\mathbf{T}\cdot\mathbf{AB} = |\mathbf{T}||\mathbf{AB}|\cos\theta = -33.32 \text{ N}\cdot\text{m}$$

The magnitude of $|\mathbf{AB}| = 0.4$ m was given but we need to calculate the magnitude of $|\mathbf{T}|$,

$$|\mathbf{T}| = \sqrt{(-50)^2 + 80^2 + 40^2} = 102.47 \text{ N}$$

$$\begin{aligned}\theta &= \cos^{-1}\frac{\mathbf{T}\cdot\mathbf{AB}}{|\mathbf{T}||\mathbf{AB}|}\\ &= \cos^{-1}\frac{-33.32 \text{ N}\cdot\text{m}}{(102.47 \text{ N})\,(0.4 \text{ m})}\\ &= 144.38°\end{aligned}$$

Note that $\theta > 90°$ correctly corresponds to the negative dot product and indicates that the two vectors generally oppose each other.

(c) Find the scalar projection of the the cable tension $\mathbf{T}$ onto the anchor $\mathbf{AB}$.

Answer.

$$||\operatorname{proj}_{\mathbf{AB}}\mathbf{T}|| = -83.30 \text{ N}$$

Solution 1. Recall from Subsection 2.7.4 that the scalar projection is the scalar component of one vector in the direction of another, in other words, how much of one vector is parallel to another. This is one of the most direct and practical applications of the dot product.

The dot product of $\mathbf{T}$ with $\mathbf{AB}$ gives the product of the length of the anchor AB and the scalar projection of the tension in the direction of the anchor.

$$\mathbf{T}\cdot\mathbf{AB} = AB\,(T\cos\theta) = AB\|\operatorname{proj}_{\mathbf{AB}}\mathbf{T}\|$$

and has units of N-m.

To find the projection, divide the dot product by the magnitude of $\mathbf{AB}$

$$\begin{aligned}\|\operatorname{proj}_{\mathbf{AB}}\mathbf{T}\| &= \frac{\mathbf{T}\cdot\mathbf{AB}}{AB} = \frac{-33.32 \text{ N}\cancel{\text{m}}}{0.4\ \cancel{\text{m}}}\\ &= -83.30 \text{ N}\end{aligned}$$

Solution 2. Alternately, you can apply (2.7.10) and calculate the calculate the dot product of force vector $\mathbf{T}$ with the unit vector $\widehat{\mathbf{AB}}$.

First, find the unit vector

$$\widehat{\mathbf{AB}} = \frac{\mathbf{AB}}{AB} = \frac{\langle 0.3464, -0.2, 0\rangle \text{ m}}{0.4 \text{ m}} = \langle 0.866, -0.5, 0\rangle$$

Or, since $\mathbf{AB}$ is in the xy plane with its direction defined by the $\alpha = 30°$, the unit vector $\widehat{\mathbf{AB}}$ is found

$$\widehat{\mathbf{AB}} = \langle\cos 30°, -\sin 30°, 0\rangle = \langle 0.866, -0.5, 0\rangle.$$

Then find the projection

$$\begin{aligned}\|\operatorname{proj}_{\mathbf{AB}}\mathbf{T}\| &= \mathbf{T}\cdot\widehat{\mathbf{AB}}\\ &= (-50 \text{ N})\,(0.866) + (80 \text{ N})\,(-0.5) + (40 \text{ N})\,(0)\\ &= -83.30 \text{ N}\end{aligned}$$

(d) Find the vector projection of the cable tension $\mathbf{T}$ onto the anchor $\mathbf{AB}$.

Answer.

$$\operatorname{proj}_{\mathbf{AB}} \mathbf{T} = \langle -72.14, 41.65, 0 \rangle \text{ N}$$

Solution. The vector projection is the scalar projection value multiplied by a unit direction vector to turn it a vector. So we multiply the scalar projection with the unit vector of $\widehat{\mathbf{AB}}$ to compute the vector projection of T onto AB.

$$\begin{aligned}\operatorname{proj}_{\mathbf{AB}} \mathbf{T} &= \|\operatorname{proj}_{\mathbf{AB}} \mathbf{T}\| \cdot \widehat{\mathbf{AB}}\\ &= 83.301 \text{ N} \left(\langle 0.866, -0.5, 0 \rangle\right)\\ &= \langle -72.14, 41.65, 0 \rangle \text{ N}\end{aligned}$$

(e) Find the vector portion of cable tension $\mathbf{T}$ perpendicular to the anchor $\mathbf{AB}$.

Answer.

$$\mathbf{T}_{\perp}\mathbf{AB} = \langle 22.14, 38.35, 40 \rangle \text{ N}$$

Solution. Recall that a two-dimensional vector can be represented by the sum of two perpendicular components. In the same way, a right triangle can be represented by a vector along the hypotenuse equal to the sum of the two right-triangle sides.

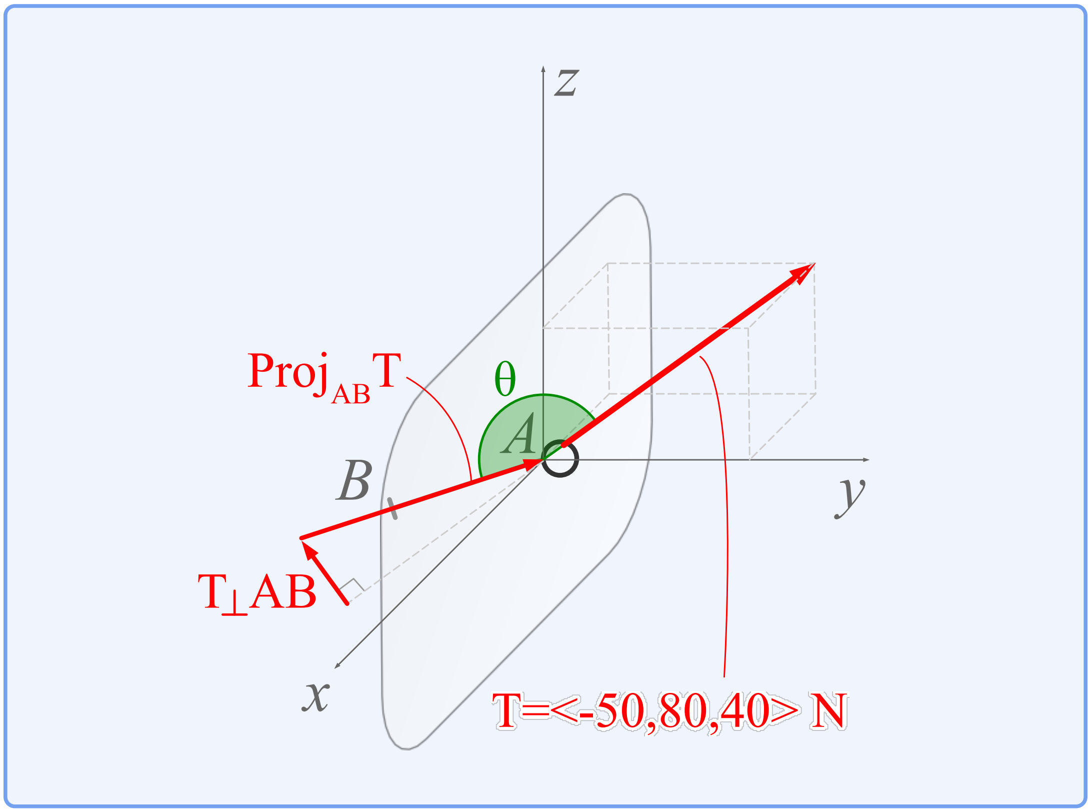
Thus, any vector can be divided into two vectors parallel and perpendicular to another line. The vector projection $\operatorname{proj}_{\mathbf{AB}} \mathbf{T}$, from Part (d), is the portion of $\mathbf{T}$ parallel to $\mathbf{AB}$. So the sum of $\mathbf{T}$ can be expressed as the parallel and perpendicular terms:

$$\mathbf{T} = \operatorname{proj}_{\mathbf{AB}} \mathbf{T} + (\mathbf{T} \perp \mathbf{AB})$$

We want to find the part of $\mathbf{T}$ perpendicular to $\mathbf{AB}$, so we can rearrange the equation to find:

$$\begin{aligned}\mathbf{T} \perp \mathbf{AB} &= \mathbf{T} - \operatorname{proj}_{\mathbf{AB}} \mathbf{T}\\ &= \langle -50, 80, 40 \rangle - \langle -72.14, 41.65, 0 \rangle\\ &= \langle 22.14, 38.35, 40 \rangle \text{ N}\end{aligned}$$

Nice effort if you worked through all the parts of this example. Graphically the results for parts (b), (d), and (e) are shown in this diagram.



2.8 Cross Products

Key Questions

- How is a cross product different than a dot product?
- What is a determinant?
- What defines a right-handed Cartesian coordinate system?
- How do you use the cross-product circle to find the cross product of two unit vectors?

The vector **cross product** is a multipliation operation applied to two vectors which produces a third mutually perpendicular vector as a result. It's sometimes called the **vector product** to emphasize this and to distinguish it from the dot product which produces a scalar result. The $\times$ symbol is used to indicate this operation.

Cross products are used in mechanics to find the moment of a force about a point.

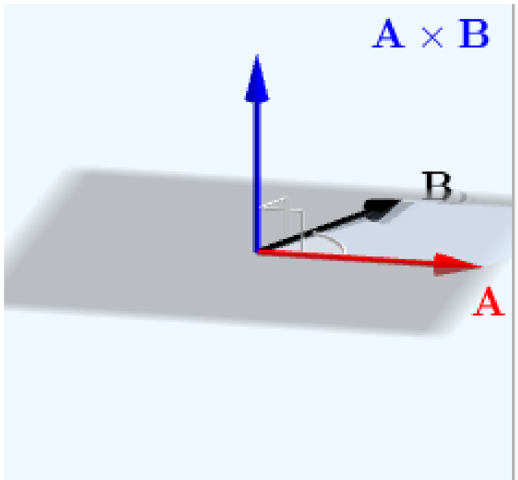


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Figure 2.8.1 Direction of a cross product.

The cross product is a vector multiplication process defined by

$$\mathbf{A} \times \mathbf{B} = A\ B \sin\theta\ \hat{\mathbf{u}}. \tag{2.8.1}$$

The result is a vector mutually perpendicular to both with a sense determined by the right-hand rule. If $\mathbf{A}$ and $\mathbf{B}$ are in the xy plane, this is

$$\mathbf{A} \times \mathbf{B} = (A_y B_x - A_x B_y)\ \mathbf{k}. \tag{2.8.2}$$

The operation is not commutative, in fact reversing the order introduces a negative sign.

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}.$$

The magnitude of the cross product is the product of the perpendicular component of $\mathbf{A}$ with the magnitude of $\mathbf{B}$,. This is the area of the parallelogram formed by vectors $\mathbf{A}$ and $\mathbf{B}$. The magnitude of the cross product is zero if $\mathbf{A}$ and $\mathbf{B}$ are parallel, and it is maximum when they are perpendicular. The magnitude of the cross product of two perpendicular *unit* vectors is one.

Notice that the cross product equation are similar to the dot product, except that sin is used rather than cos and the product includes a unit vector $\hat{\mathbf{u}}$ making the result a vector. This unit vector $\hat{\mathbf{u}}$ is simple to find in a two-dimensional problem as it will always be perpendicular to the page, but for three-dimensional cross products a vector determinant is used, as discussed in Subsection 2.8.3.

2.8.1 Direction of the Vector Cross Product

The direction of a cross product is determined by the **right-hand rule.** There are two ways to apply the right-hand rule, the three-finger method, and the point-and-curl method. You don't need both, but you will need to master at least one to find the direction of cross products.

The three-finger method uses the fact that your extended index finger, middle finger, and thumb are all roughly mutually perpendicular. If you align your index finger with the first vector and your middle finger with the second, then your thumb will point in the direction of the cross product. Alternately, if you align your thumb with the first vector and your index finger with the second, your middle finger will point in the direction of the cross-product.

(a) Technique 1 **(b)** Technique 2

Figure 2.8.2 Right-hand rule using three fingers.

The point-and-curl method involves placing your right hand flat with your fingertips pointing in the direction of the the first vector. Then rotate your hand until the second vector is can curl your fingers around your thumb. In this position, your thumb defines the direction of the cross product.

(a) Step one **(b)** Step one

Figure 2.8.3 Right-hand rule using the point-and-curl technique.

2.8.2 Cross Product of Unit Vectors

The Figure 2.8.4(a) demonstrates how you apply these techniques to find the cross product of $\mathbf{i} \times \mathbf{j}$. Assuming the x axis points right and the y axis points up, the cross product points in the positive z direction. Recalling that the magnitude of the cross product of two peperpedicular unit vectors is one, we conclude that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

(a) Using the three-finger method. **(b)** Using the point-and-curl method.

Figure 2.8.4 Crossing $\mathbf{i}$ into $\mathbf{j}$ to get $\mathbf{k}$.

Similarly, the cross products of the other pairs of vectors are:

$$\begin{array}{lll}
\mathbf{i}\times\mathbf{i}=0 & \mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{i}\times\mathbf{k}=-\mathbf{j}\\
\mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{j}\times\mathbf{j}=0 & \mathbf{j}\times\mathbf{k}=\mathbf{i}\\
\mathbf{k}\times\mathbf{i}=\mathbf{j} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{k}\times\mathbf{k}=0
\end{array}$$

An alternate way to remember this is to use the cross-product circle shown. For example when you cross **i** with **j** you are going in the positive (counter-clockwise) direction around the blue inner circle and thus the answer is $+\mathbf{k}$. But when you cross **j** into **i** you go in the negative (clockwise) direction around the circle and thus get a $-\mathbf{k}$. Remember that the order of cross products matter. If you put the vectors in the wrong order you will introduce a sign error.

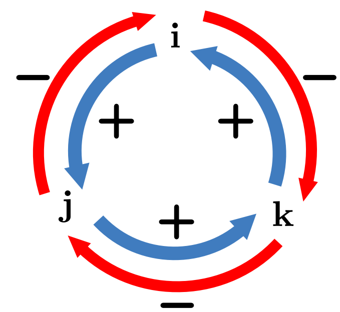


Figure 2.8.5 Unit vector cross product circle.

If you have any negative unit vectors it is easiest to pull out the negative signs before you take the cross product, like the following.

$$-\mathbf{j}\times\mathbf{i}=(-1)\,(\mathbf{j}\times\mathbf{i})=(-1)(-\mathbf{k})=+\mathbf{k}$$

2.8.3 Cross Product of Arbitrary Vectors

The cross product of two arbitrary three-dimensional vectors can be calculated by evaluating the **determinant** of this 3×3 matrix.

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \tag{2.8.3}$$

Here, the first row contains the unit vectors, the second row contains the components of **A**, and the third row, the components of **B**. The determinant of this 3×3 matrix is evaluated using the method of cofactors, as follows

$$\mathbf{A}\times\mathbf{B}=+\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}\,\mathbf{i}-\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}\,\mathbf{j}+\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}\,\mathbf{k}. \tag{2.8.4}$$

Each term contains a 2×2 determinant which is evaluated with the formula

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc. \tag{2.8.5}$$

After simplifying, the resulting formula for a three-dimensional cross product is

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\, \mathbf{i} - (A_x B_z - A_z B_x)\, \mathbf{j} + (A_x B_y - A_y B_x)\, \mathbf{k}. \tag{2.8.6}$$

In practice, the easiest way to remember this equation is to use the augmented determinant below, where the first two columns have been copied and placed after the determinant. The cross product is then calculated by adding the product of the red diagonals and subtracting the product of blue diagonals. The result is identical to (2.8.6).

$$\mathbf{A} \times \mathbf{B} = \left|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{matrix}\right| \begin{matrix} \mathbf{i} & \mathbf{j} \\ A_x & A_y \\ B_x & B_y \end{matrix}$$

Figure 2.8.6 Augmented determinant

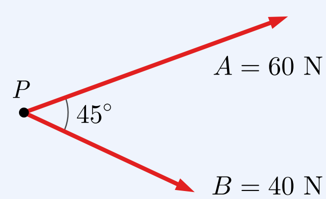
In two dimensions, vectors **A** and **B** have no z components, so (2.8.6) reduces to

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (A_x B_y - A_y B_x)\, \mathbf{k}. \tag{2.8.7}$$

This equation produces the same result as equation (2.8.1) and you may use it if it is more convenient.

Example 2.8.7 2D Cross Product.

The two vectors **A** and **B** shown lie in the xy plane. Determine the cross product $\mathbf{A} \times \mathbf{B}$.



Solution 1. In this solution we will apply equation (2.8.1).

$$\mathbf{A} \times \mathbf{B} = A\, B \sin\theta\, \hat{\mathbf{u}}$$

The direction of the cross product is determined by applying the right-hand rule. With the right hand, rotating **A** towards **B** we find that our thumb points into the xy plane, so the direction of $\hat{\mathbf{u}}$ is $-\mathbf{k}$.

$$\mathbf{A} \times \mathbf{B} = (60\text{ N})(40\text{ N}) \sin 45°(-\mathbf{k})$$

$$= 1,697 \text{ N}^2 \ (-\mathbf{k})$$
$$= -1,697 \text{ N}^2 \ \mathbf{k}$$

Solution 2. In this solution we will use (2.8.7).
First, establish a coordinate system with the origin P and with the x axis aligned with $\mathbf{A}$, then find the rectangular components and apply the cross product equation.

$$\begin{aligned} A_x &= 60 \text{ N} & A_y &= 0 \text{ N} \\ B_x &= 40 \text{ N} \cos 45° & B_y &= -40 \text{ N} \sin 45° \\ &= 28.28 \text{ N} & &= -28.28 \text{ N} \end{aligned}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x B_y - A_y B_x)\mathbf{k} \\ &= (60)(-28.28) - (0)(28.28) \text{ N}^2\mathbf{k} \\ &= -1697 \text{ N}^2 \ \mathbf{k} \end{aligned}$$

Example 2.8.8 3D Cross Product.

Find the cross product of $\mathbf{A} = \langle 2, 4, -1 \rangle$ m and $\mathbf{B} = \langle 10, 25, 20 \rangle$ N.
Here, we are crossing a *distance* $\mathbf{A}$ and with a *force* $\mathbf{B}$. This calculation is equivalent to finding the moment about a point P caused by force $\mathbf{B}$ acting distance $\mathbf{A}$ from P. You will learn about moments in Chapter 4.

Solution 1. To solve, set up the augmented determinant and evaluate it by adding the left-to-right diagonals and subtracting the right-to-left diagonals using equation (2.8.6).

$$\mathbf{A} \times \mathbf{B} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 2 & 4 & -1 & 2 & 4 \\ 10 & 25 & 20 & 10 & 25 \end{array} \right.$$

$$\begin{aligned} &= (4 \cdot 20) \ \mathbf{i} + (-1 \cdot 10) \ \mathbf{j} + (2 \cdot 25) \ \mathbf{k} - (4 \cdot 10) \ \mathbf{k} - (-1 \cdot 25) \ \mathbf{i} - (2 \cdot 20) \ \mathbf{j} \\ &= (80 + 25) \ \mathbf{i} + (-10 - 40) \ \mathbf{j} + (50 - 40) \ \mathbf{k} \\ &= \langle 105, -50, 10 \rangle \ \text{N}\cdot\text{m} \end{aligned}$$

Thus, the force $\mathbf{B}$ creates a three-dimensional rotational moment equal to $\langle 105, -50, 10 \rangle$ N·m.

Solution 2. Calculating three-dimensional cross products by hand is tedious and error-prone. Whenever you can, you should use technology to do the grunt work for you and focus on the meaning of the results. In this solution, we will use an embedded Sage calculator to calculate the cross product. This same calculator can be used to do other problems.

Given:

$$\mathbf{A} = \langle 2, 4, -1 \rangle \text{ m}$$
$$\mathbf{B} = \langle 10, 25, 20 \rangle \text{ N}.$$

A and **B** are defined in the first two lines, and `A.cross_product(B)` is the expression to be evaluated. Click `Evaluate` to see the result. You'll have to work out the correct units for yourself.

```
A = vector([2, 4, -1]);
B = vector([10, 25, 20]);
A.cross_product(B)
```

```
(-105, -50, 10)
```

Try changing the third line to `B.cross_product(A)`. What changes?

2.9 Exercises (Ch. 2)

Vectors	0/140
Rectangular components: P to R	0/20 Not attempted
Rectangular components: R to P	0/20 Not attempted
Find magnitude and direction of three vectors	0/60 Not attempted
Parallelogram Rule	0/20 Not attempted
Arbitrary components	0/20 Not attempted

Vector Addition	0/200
Tip-to-tail method	0/20 Not attempted
Parallelogram rule	0/30 Not attempted
Summing scalar components	0/60 Not attempted
Vector addition of three forces	0/20 Not attempted
Vector addition by summing scalar components	0/40 Not attempted
Description only	0/30 Not attempted

Created using Numbas, developed by Newcastle University.

Chapter 3

Equilibrium of Particles

3.1 Equilibrium

Engineering statics is the study of rigid bodies in equilibrium so it's appropriate to begin by defining what we mean by *rigid bodies* and what we mean by *equilibrium*.

A **body** is an object, possibly made up of many parts, which may be examined as a unit. In statics, we consider the forces acting on the object as a whole and also examine it in greater detail by studying each of its parts, which are bodies in their own right. The choice of the body is an engineering decision based on what we are interested in finding out. We might, for example, consider an entire high-rise building as a body for the purpose of designing the building's foundation, and later consider each column and beam of the structure to ensure that they are strong enough to perform their individual roles.

A **rigid body** is a body that doesn't deform under load, that is to say, an object which doesn't bend, stretch, or twist when forces are applied to it. It is an idealization or approximation because no objects in the real world behave this way; however, this simplification still produces valuable information. You will drop the rigid body assumption and study deformation, stress, and strain in a later course called Strength of Materials or Mechanics of Materials. In that course, you will perform analysis of non-rigid bodies, but each problem you do *there* will begin with the rigid body analysis you will learn to do *here*.

A body in **equilibrium** is not accelerating. As you learned in physics, acceleration is velocity's time rate of change and is a vector quantity. For linear motion,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}.$$

For an object in equilibrium $\mathbf{a} = 0$ which implies that the body is either stationary or moving with a constant velocity

$$\mathbf{a} = 0 \implies \begin{cases} \mathbf{v} = 0 \\ \mathbf{v} = C \end{cases} .$$

The acceleration of an object is related to the net force acting on it by **Newton's Second Law**

$$\Sigma \mathbf{F} = m\mathbf{a}.$$

So for the special case of static equilibrium Newton's Law becomes

$$\Sigma \mathbf{F} = 0. \tag{3.1.1}$$

This simple equation is one of the two foundations of engineering statics.

There are several ways to think about this equation. Reading it from left to right it says that *if all the forces acting on a body sum to zero, then the body will be in equilibrium.* If you read it from right to left it says that *if a body is in equilibrium, then all the forces acting on the body must sum to zero.* Both interpretations are equally valid but we will be using the second one more often. In a typical problem equilibrium of a body implies that the forces sum to zero, and we use that fact to find the unknown forces which make it so. Remember that we are talking about vector addition here, so the sums of the forces must be calculated using the rules of vector addition; you won't get correct answers if you can't add vectors!

We'll be using all of the different vector addition techniques introduced in Section 2.6, which may lead to some confusion. It doesn't matter, mathematically, which technique you use but part of the challenge and reward of statics is learning to select the best tool for the job at hand; to select the simplest, easiest, fastest, or clearest way to get to the solution. You'll do best in this course if can use multiple approaches to solve the same problem.

In Chapter 5 we will add another requirement for equilibrium, namely equilibrium equation (5.3.2) which says the forces which cause rotational motion and angular acceleration $\boldsymbol{\alpha}$ also must sum to zero, but for the problems of this chapter the only condition we'll need for equilibrium is $\Sigma \mathbf{F} = 0$.

3.2 Particles

We'll begin our study of Equilibrium with the simplest possible object in the simplest possible situation — a **particle** in a **one-dimensional coordinate system**. Also, in this chapter and the next all forces will be represented as **concentrated forces**. In later sections, we will address more complicated situations, higher dimensions, and distributed forces, but beginning with very simple situations will help you to develop engineering sense and problem-solving skills which will be useful later.

The defining characteristic of a particle is that all forces that act on it are coincident[1] or concurrent[2], not that it is small. Forces are coincident if they have the same line of action, and concurrent if they intersect at a point. The moon, earth and sun can all be treated as particles, but we probably won't encounter

[1] Two lines are coincident when one lies on top of the other.

[2] Two or more lines are concurrent if they intersect at a single point.

them in statics since they're not in equilibrium. Forces are coincident/concurrent if their lines of action all intersect at a single, common point. Two or more forces are also considered concurrent if they share the same line of action. One practical consequence of this is that particles are never subjected to forces that cause rotation. So a see-saw, for example, is not a particle because the weights of the children tend [3] to cause rotation.

Another consequence of concurrent forces is that Equation (3.1.1) is the only equilibrium equation that applies. This vector equation can be used to solve for a maximum of one unknown per dimension. If you find yourself trying to solve a two-dimensional particle equilibrium problem and you are seeking more than two unknowns, it's likely that you have missed something and need to re-read the question.

Another simplification we will be making is to treat all forces as **concentrated**. Concentrated forces act at a single point, have a well-defined line of action, and can be represented with an arrow — in other words, they are vectors. Real forces don't actually act at a single mathematical point but concentrating them is intuitive and will be justified in a later chapter **??**. You're already familiar with the concept if you have ever placed all the weight of an object at its center of gravity.

3.3 1D Particle Equilibrium

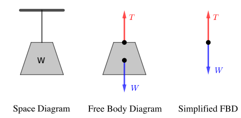
3.3.1 A simple case

Consider the weight suspended by a rope shown in Figure 3.3.1. Diagrams of this type are called **space diagrams**; they show the objects as they exist in space.

In mechanics we are interested in studying the forces acting on objects and in this course, the objects will be in equilibrium. The best way to do this is to draw a diagram that focuses on the forces acting on the object, not the mechanisms that hold it in place. We call this type of diagram a free-body diagram because it shows the object disconnected or freed from its supporting mechanisms. You can see the free-body diagram for this situation by moving the slider in the interactive to position two. This shows that there are two forces acting on the object; the force of the rope holding it up, and the weight of the object which is trying to pull it to earth, which we treat as acting at its center of gravity.

[3] We say "tend to cause rotation" because in a static's context, all objects are static — so no actual rotation occurs.

The actual shape of the weight is not important to us, so it can simply be represented with a dot, as shown when the view control is in position three. The forces have been slid along their common line of action until they both act on the dot, which is an example of an equivalent transformation called the “Principle of Transmissibility.” This diagram in view three is completely sufficient for this situation.



Stand-alone Embed

Figure 3.3.1 A suspended weight

Drawing free-body diagrams can be surprisingly tricky. The reason for this is that you must identify all the forces acting on the object and correctly represent them on the free-body diagram. If you fail to account for all the forces, include additional ones, or represent them incorrectly, your analysis will surely be wrong.

So what kind of analysis can we do here? Admittedly not much. We can find the tension in the rope caused by a particular weight and use it to select an appropriately strong rope, or we can determine the maximum weight a particular rope can safely support.

The actual analysis is so trivial that you’ve probably already done it in your head, nevertheless several ways to approach it will be shown next.

In the vector approach we will use the equation of equilibrium.

Example 3.3.2 1D Vector Addition.

Find the relationship between the tension in the rope and the suspended weight for the system of Figure 3.3.1.

Solution.

The free-body diagram shows two forces acting on the particle, and since the particle is in equilibrium they must add to zero.

$$\begin{aligned} \Sigma\mathbf{F} &= 0\\ \mathbf{T}+\mathbf{W} &= 0\\ \mathbf{T} &= -\mathbf{W} \end{aligned}$$

We conclude that force **T** is equal and opposite to **W**, that is, since the weight is acting down, the rope acts with the same magnitude but up. **Tension** is the magnitude of the rope’s force. Recall that the magnitude of a vector is always a positive scalar. We use normal (non-bold) typefaces or absolute value bars surrounding a vector to indicate its magnitude. For any force **F**,

$$F = |\mathbf{F}|.$$

To find how the tension is related to **W**, take the absolute value of both sides

$$|\mathbf{T}| = |-\mathbf{W}|$$
$$T = W$$

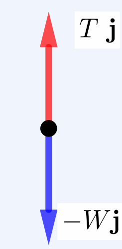
We can also formulate this example in terms of unit vectors. Recall that **j** is the unit vector that points up. It has a magnitude of one with no units associated. So in terms of unit vector **j**, $\mathbf{T} = T\mathbf{j}$ and $\mathbf{W} = -W\mathbf{j}$.

Example 3.3.3 1D Vector Addition using unit vectors.

Find the relation between the tension T and weight W for the system of Figure 3.3.1 using unit vectors.

Solution.

Express the forces in terms of their magnitudes and the unit vector **j** then proceed as before,



$$\begin{aligned}\Sigma\mathbf{F} &= 0\\ \mathbf{T} + \mathbf{W} &= 0\\ T\mathbf{j} + W(-\mathbf{j}) &= 0\\ T\cancel{\mathbf{j}} &= W\cancel{\mathbf{j}}\\ T &= W\end{aligned}$$

In the previous example, the unit vector **j** completely dropped out of the equation leaving only the coefficients of **j**. This will be the case whenever you add vectors which all act along the same line of action.

The coefficients of **i**, **j**, and **k** are known as the **scalar components**. A scalar component times the associated unit vector is a force vector.

When you use scalar components, the forces are represented by scalar values and the equilibrium equations are solved using normal algebraic addition rather than vector addition. This leads to a slight simplification of the solution as shown in the next example.

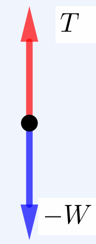
Example 3.3.4 1D Vector Addition using scalar components.

Find the relation between the tension T and weight W for the system of Figure 3.3.1 using scalar components.

Solution.

The forces in this problem are $\mathbf{W} = -W\,\mathbf{j}$ and $\mathbf{T} = T\,\mathbf{j}$, so the corresponding scalar components are

$$W_y = -W \qquad\qquad T_y = T.$$



Adding scalar components gives,

$$\begin{aligned} \Sigma F_y &= 0 \\ W_y + T_y &= 0 \\ -W + T &= 0 \\ T &= W \end{aligned}$$

Unsurprisingly, we get the same result.

3.3.2 Scalar Components

The scalar component of a vector is a *signed* number which indicates the vector's **magnitude** and **sense**, and is usually identified by a symbol with a subscript which indicates the line of action of the vector.

So for example, $F_x = 10$ N is a scalar component. We can tell it's not a vector because it F_x is not bold. 10 N is the magnitude of the associated vector; the subscript x indicates that the force acts "in the x direction," in other words it acts on a line of action which is parallel to the x axis; and the (implied) positive sign means that the vector points towards the positive end of the x axis — towards positive infinity. So a scalar component, while not a vector, contains all the information necessary to completely describe and draw the corresponding vector. Be careful not to confuse scalar components with vector magnitudes. A force with a magnitude of 10 N can point in any direction, but can never have a negative magnitude.

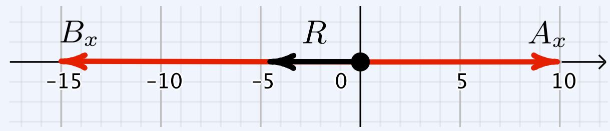
Scalar components can be added together algebraically, but only if they act "in the same direction." It makes no sense to add F_x to F_y. If that's what you want to do, first you must convert the scalar components to vectors, then add them according to the rules of vector addition.

Example 3.3.5 1D Scalar Addition.

If $A_x = 10$ lb and $B_x = -15$ lb, find the magnitude and direction of their resultant $\mathbf{R}$.

Solution. Start by sketching the two forces. The subscripts indicate the line of action of the force, and the sign indicates the direction along the line of action. A negative B_x points towards the negative end of the x axis.

$$\begin{aligned} R &= A_x + B_x \\ &= 10 \text{ lb} + -15 \text{ lb} \\ &= -5 \text{ lb} \end{aligned}$$

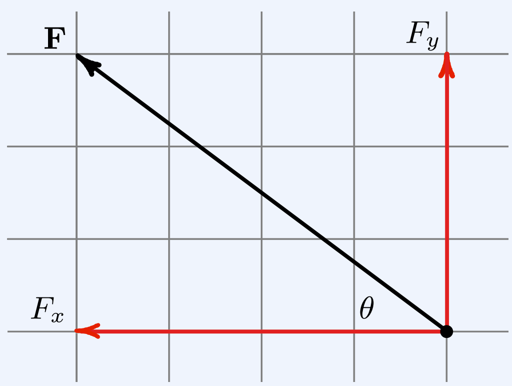


R is the scalar component of the resultant $\mathbf{R}$.
The negative sign on the result indicates that the resultant force acts to the left.

Example 3.3.6 2D Scalar Addition.

If $F_x = -40$ N and $F_y = 30$ N, find the magnitude and direction of their resultant $\mathbf{F}$.

Solution. In this example the scalar components have different subscripts indicating that they act along different lines of action, and this must be accounted for when they are added together.
Make a sketch of the two vectors and add them using the parallelogram rule. In this case, the parallelogram is a rectangle, so right-triangle trig is appropriate.



$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(-40 \text{ N})^2 + (30 \text{ N})^2} \\ &= 50 \text{ N} \\ \theta &= \tan^{-1}\left|\frac{F_y}{F_x}\right| \\ &= \tan^{-1}\left|\frac{30 \text{ N}}{-40 \text{ N}}\right| \\ &= 36.9^\circ \end{aligned}$$

θ is measured from the negative x axis. The direction of $\mathbf{F}$ from the positive

x axis is $(180° - \theta) = 143.1°$, so

$$\mathbf{F} = 50 \text{ N at } 143.1°\measuredangle$$

3.3.3 Two-force Bodies

As you might expect from the name, a **two-force body** is a body with two forces acting on it, like the weight just discussed. As we just saw, in order for a two-force body to be in equilibrium the two forces must add to zero. There are only three possible ways that this can happen:

The two forces must either

- share the same line of action, have the same magnitude, and point away from each other, or
- share the same line of action, have the same magnitude, and point towards each other, or
- both forces have zero magnitude.

When two forces have the same magnitude but act in diametrically opposite directions, we say that they are **equal-and-opposite**. When equal and opposite forces act on an object and they point towards each other we say that the object is **in compression**, when they point away from each other the object is **in tension**. Tension and compression describe the internal state of the object.

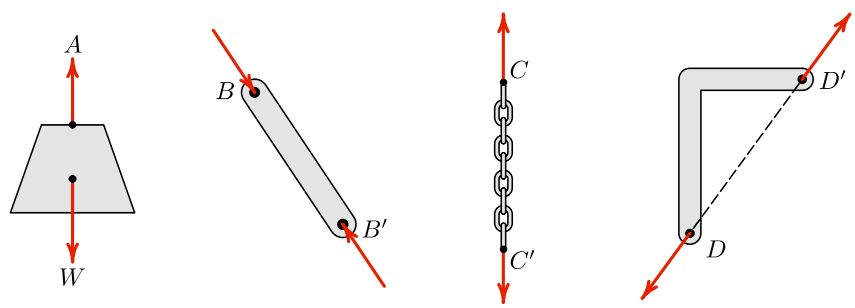


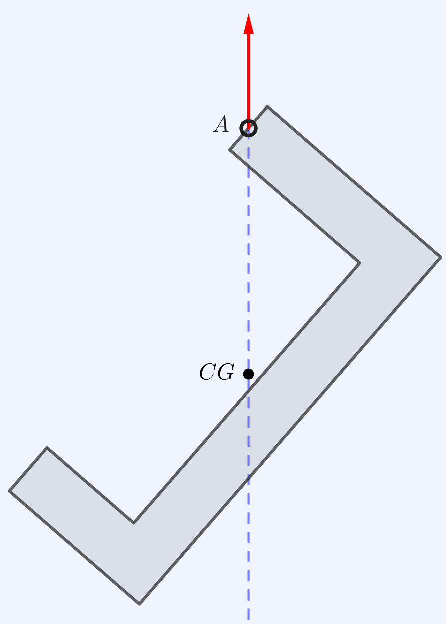
Figure 3.3.7 Examples of two-force bodies

Did you notice that last three examples in Figure 3.3.7 did not include the object's weight? These are simplifictions that ignore the object's weight to make them two-force bodies. If the object's weight was included, it would be a *three-force* body. This approximation is justifiable when the object's weight is small in comparison with the tensile or compressive forces. In this case, we say that the weight is **negligable**, i.e. small enough to neglect. Also note that all these examples show single forces acting at each point. If several forces act at a point, they should be combined into a single resultant force acting there.

Two force bodies appear frequently in multipart structures and machines which will be covered in Chapter 6. Some examples of two force bodies are struts and linkages, ropes, cables and guy wires, and springs.

Thinking Deeper 3.3.8 Locating the Center of Gravity.

As we will see in Chapter 7, the center of gravity is the point where all the weight of an object can be considered to be concentrated.
An object suspended by a cable or a frictionless pin is a two-force body. When hung freely it will naturally rotate until its center of gravity lies directly beneath the support point to ensure that the lifting force and the weight share the same line of action. This means that the center of gravity of an object can be found by suspending it from several different points, and noting intersection of lines drawn straight down from the hook (like a plumb bob).
In practical terms, to safely lift a heavy object with a chain fall or crane, you must always ensure that the hook is directly above the center of gravity before hoisting the load. The load will be unstable if lifted from any other point.



Standalone
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Example 3.3.9 Tug of War.

Marines and Airmen at Goodfellow Air Force Base are competing in a tug of war and have reached a stalemate. The Marines are pulling with a force of 1500 lb. How hard are the Airmen pulling? What is the tension in the rope?

This is a simple question, but students often get it wrong at first.

Solution.

1. *Assumptions.*

   A free-body diagram of the rope is shown.

   Figure 3.3.10

   We'll solve this with scalar components because there's no need for the additional complexity of the vector approaches in this simple situation.

   We'll align the x axis with the rope with positive to the right as usual to establish a coordinate system.

   Assume that the pull of each team can be represented by a single force. Let force M be supplied by the Marines and force A by the Airmen; call the tension in the rope T.

   Assume that the weight of the rope is negligible; then the rope can be considered a particle because both forces lie along same line of action.

2. *Givens.*

   $M = 1500$ lb.

3. *Procedure.*

   Since they're stalemated we know that the rope is in equilibrium.

   Applying the equation of equilibrium gives:

$$\Sigma F_x = 0$$
$$-M + A = 0$$

$$\begin{aligned} A &= M \\ &= 1500 \text{ lb} \end{aligned}$$

We find out that both teams pull with the same force. This was probably obvious without drawing the free-body diagram or solving the equilibrium equation.

It may seem equally obvious that if both teams are pulling with 1500 lb in opposite directions that the tension in the rope must be 3000 lb. This is wrong however.

The tension in the rope T is an example of an internal force and in order to learn its magnitude we need a free-body diagram which includes force T. To expose the internal force we take an imaginary cut through the rope and draw (or imagine) a free-body diagram of either half of the rope.

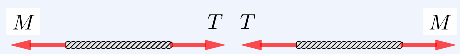
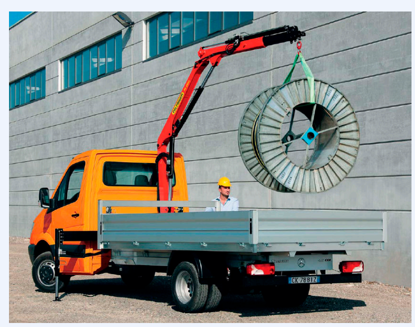


Figure 3.3.11

The correct answer is easily seen to be $T = A = M = 1500$ lb.

Example 3.3.12 Hanging Weight.

The wire spool being lifted into the truck consists of 750 m of three strand medium voltage (5 kV) 1/0 AWG electrical power cable with a 195 amp capacity at 90°C, weighing 927 kg/km, on a 350 kg steel reel.
How much weight is supported by the hook and high tension polymer lifting sling?



Solution. The entire weight of the wire and the spool is supported by the hook and sling.
Remember that weight is not mass and mass is not force. The total weight is found by multiplying the total mass by the gravitational constant g.

$$\begin{aligned} W &= mg \\ &= (m_w + m_s)\, g \\ &= ((0.75 \text{ km})(927 \text{ kg/km}) + 350 \text{ kg})\, g \\ &= (1045 \text{ kg})(9.81 \text{ m/s}^2) \end{aligned}$$

$$= 10300 \text{ N}$$

Question 3.3.13

How can we apply the principles of mechanics in the two previous examples if the rope and the sling are clearly not "rigid bodies?"

Answer. They are not rigid, but they are inextensible and in tension. Under these conditions they don't change shape, so we can treat them as rigid. If the force were to change direction and put either into compression, our assumptions and analysis would fail. That is why "tug of war" involves pulling and not pushing.

3.3.4 General Procedure

The general procedure for solving one, two, or three-dimensional particle equilibrium problems is essentially the same. Start with the 5-step method for creating a free-body diagram and solve for your unknowns using your equilibrium equations.

Draw a Free-Body Diagram

1. *Select and isolate the particle.* The "free-body" in free-body diagram means that a concurrent force particle or connection must be isolated from the supports physically holding it in place. This means creating a separate free-body diagram from your problem sketch.

2. *Establish a coordinate system.* This step is simple for one-dimensional problems: just label a positive direction for the forces.

3. *Identify all loads.* Include force vectors on your free-body diagram representing each applied load pushing or pulling the body, in addition to the body's weight, if it is non-negligible. Every vector should have a descriptive variable name and a clear arrowhead indicating its direction.

4. *Identify all reactions.* Reactions represent the resistance of the physical supports you cut away by isolating the body in step 1. All particle supports are two-force members that result in tension or compression forces. Label each reaction with a descriptive variable name and a clear arrowhead.

5. *Label the diagram.* Verify that every force is labeled with either a value or a symbolic name if the value is unknown. Your final free-body diagram should be a stand-alone presentation and is the basis of your equilibrium equations.

Create and Solve Equilibrium Equations

1. *Write the equilibrium equation.* Now represent your free-body diagram as an equilibrium equation. Your computation should start with the governing equation, like $\Sigma\mathbf{F} = 0$.

2. *Solve for unknown.* Use algebra to simplify the equilibrium equation and solve for the unknown value. Write the unit of your answer. All answers in engineering have units unless you *prove* that they don't. Finally, underline or box your answers.

3. *Check your work.* Do the results seem reasonable given the situation? Have you included appropriate units?

3.4 2D Particle Equilibrium

3.4.1 Introduction

In this section we will study situations where everything of importance occurs in a two-dimensional plane and the third dimension is not involved. Studying two-dimensional problems is worthwhile because they illustrate all the important principles of engineering statics while being easier to visualize and less mathematically complex.

We will normally work in the "plane of the page," that is, a two-dimensional Cartesian plane with a horizontal x axis and a vertical y axis discussed in Section 2.3 previously. This coordinate system can represent either the front, side, or top view of a system as appropriate. In some problems it may be worthwhile to **rotate the coordinate** system, that is, to establish a coordinate system where the x and y axes are not horizontal and vertical. This is usually done to simplify the mathematics by avoiding simultaneous equations.

3.4.2 General Procedure

The general procedure for solving two-dimensional particle equilibrium is a step up from solving Subsection 3.3.1, as you now need to find equilibrium in two independent directions. The major difference is that you must carefully find each independent vector component and then solve for the equilibrium in each component direction. The process follows the same five-step method for creating a free-body diagram, followed by steps to solve your equilibrium equations.

Draw a Free-Body Diagram:

1. *Select and isolate the particle.* The "free-body" in free-body diagram means that a concurrent force particle or connection must be isolated from the supports that are physically holding it in place. This means creating a separate free-body diagram from your problem sketch.

2. *Establish a coordinate system.* Draw a right-handed coordinate system to use as a reference for your equilibrium equations. Look ahead and select a coordinate system that minimizes the number of force components. This will simplify your vector algebra. The choice is technically arbitrary, but a good choice will simplify your calculations and reduce your effort.

3. *Identify all loads.* Add force vectors to your free-body diagram representing each applied load pushing or pulling the body, in addition to the body's weight, if it is non-negligible. If a force vector has a known direction, draw it. If its direction is unknown, assume one, and your later algebra will check your assumption. Every vector should have a descriptive variable name and a clear arrowhead indicating its direction.

4. *Identify all reactions.* Reactions represent the resistance of the physical supports you cut away by isolating the body in step 1. All particle supports are some type of two-force members with tension or compression reaction forces. These reactions will all be concurrent with the body loads from Step 2. Label each reaction with a descriptive variable name and a clear arrowhead. Again, if a vector's direction is unknown, just assume one.

5. *Label the diagram.* Verify that every dimension, angle, force, and moment is labeled with either a value or a symbolic name if the value is unknown. In our eyes, dimensioning is optional. Having the information needed for your calculations is helpful, but don't clutter the diagram up with unneeded details. Your final free-body diagram should be a stand-alone presentation and is the basis of your equilibrium equations.

Create and Solve Equilibrium Equations

1. *Break vectors into components.* Compute each force's x and y components using right-triangle trigonometry.

2. *Write equilibrium equations.* Now represent your free-body diagram as two equilibrium equations, $\Sigma \mathbf{F_x} = 0$ and $\Sigma \mathbf{F_y} = 0$.

3. *Count knowns and unknowns.* At this point, you should have at most two unknown values. If you have more than two, reread the problem and look for overlooked information.

4. *Solve for unknowns.* Use algebra to simplify the equilibrium equations and solve for unknowns. All answers in Statics will have units - unless you have solved for a dimensionless value, like a friction coefficient. Finally, underline or box your answers.

5. *Check your work.* If you add the components of the forces, do they add to zero? Do the results seem reasonable given the situation? Have you included appropriate units?

3.4.3 Force Triangle Method

The force triangle method applies to situations where there are (exactly) three forces acting on a particle, and no more than two unknown magnitudes or directions.

If such a particle is in equilibrium then the three forces must add to zero. Graphically, if you arrange the force vectors tip-to-tail, they will form a closed, three-sided polygon, i.e. a triangle. This is illustrated in Figure 3.4.1.

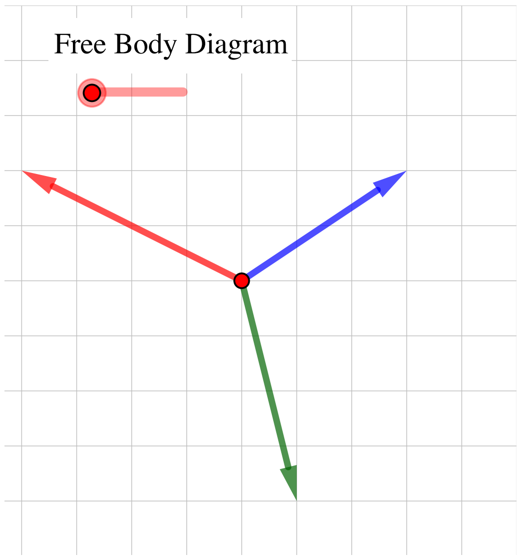


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Figure 3.4.1 Free-Body Diagram and Force Triangle

Question 3.4.2

Why do the forces always form a closed polygon?

Answer. Because their resultant is zero.

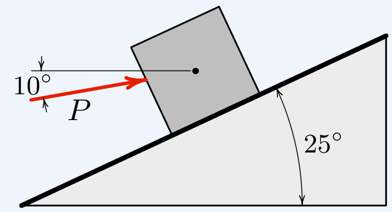
The force triangle is a graphical representation of the vector equilibrium equation (3.1.1). It can be used to solve for unknown values in multiple different ways, which will be illustrated in the next two examples. In Example 3.4.3 We will use a graphical approach to find the forces causing equilibrium, and in Example 3.4.4 we will use trigonometry to solve for the unknown forces mathematically.

In the next example we will use technology to draw a scaled diagram of the force triangle representing the equilibrium situation. We are using Geogebra[1] to make the drawing, but you could use CAD, another drawing program, or even a ruler and protractor as you prefer. Since the diagram is accurately drawn, the lengths and angles represent the magnitudes and directions of the forces which hold the particle in equilibrium.

[1] geogebra.org

Example 3.4.3 Frictionless Incline.

A force P is being applied to a 100 lb block resting on a frictionless incline as shown. Determine the magnitude and direction of force P and of the contact force on the bottom of the block.



Solution.

1. *Assumptions.*

   We must assume that the block is in equilibrium, that is, either motionless or moving at a constant velocity in order to use the equilibrium equations. We will represent the block's weight and the force between the incline and the block as concentrated forces. The force of the inclined surface on the block must act in a direction that is normal to the surface since it is frictionless and can't prevent motion along the surface.

2. *Givens.*

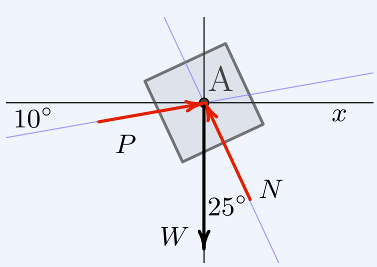
   The knowns here are the weight of the block, the direction of the applied force, and the slope of the incline. The slope of the incline provides the direction of the normal force.

   The unknown values are the magnitudes of forces P and N.

3. *Free-Body Diagram.*

   You should always begin a statics problem by drawing a free-body diagram. It allows you to think about the situation, identify knowns and unknowns, and define symbols.

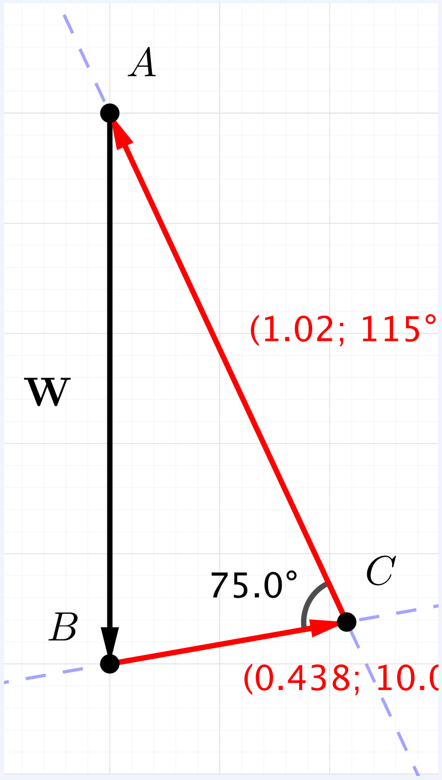
   We define three symbols, W, N, and P, representing the weight, normal force, and applied force respectively. The angles could be given symbols too, but since we know their values it isn't necessary.

   The free-body can be a quick sketch or an accurate drawing but it must show *all* the forces acting on the particle and define the symbols. In most cases, you won't know the magnitudes of all the forces, so the lengths of the vectors are just approximate.

   Notice that the force N is represented as acting 25° from the y axis, which is 90° away from the direction of the surface.

4. *Force Triangle.*



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Use the known information to carefully and accurately construct the force triangle.

(a) Start by placing point A at the origin.

(b) Draw force $\mathbf{W}$ straight down from A with a length of 1, and place point B at its tip. The length of this vector represents the weight.

(c) We know the direction of force $\mathbf{P}$ but not its magnitude. For now, just draw line BC passing through point B with an angle of 10° from the horizontal.

(d) Similarly we know force $\mathbf{N}$ acts at 25° from vertical because it is perpendicular to the inclined surface, and it will close the triangle. So draw line CA passing through point A and at a 25° angle from the y axis.

(e) Call the point where lines BC and CA intersect point C. Points A, B, and C define the force triangle.

(f) Now draw force $\mathbf{P}$ from point B to point C, and

(g) Draw force $\mathbf{N}$ from point C back to point A.

Can you prove from the geometry of the triangle that angle BCA is 75°?

5. *Results.*

   In steps 6 and 7, Geogebra tells us that `p = (0.438;10.0°)` which means force P is 0.438 units long with a direction of 10°, similarly `n = (1.02;115°)` means N is 1.02 units long at 115°. These angles are measured counter-clockwise from the positive x axis.

   These are not the answers we are looking for, but we're close. Remember that for this diagram, our scale is

   $$1 \text{ unit} = 100 \text{ lbs},$$

   so scaling the lengths of `p` and `n` by this factor gives

   $$\begin{aligned} P &= (0.438 \text{ unit})(100 \text{ lb/ unit}) \\ &= 43.8 \text{ lb at } 10°\measuredangle \\ N &= (1.02 \text{ unit})(100 \text{ lb/ unit}) \\ &= 102 \text{ lb at } 115°\measuredangle. \end{aligned}$$

   If you use technology such as Geogebra, as we did here, or CAD software to draw the force triangle, it will accurately produce the solution.

   If technology isn't available to you, such as during an exam, you can still use a ruler and protractor to draw the force triangle, but your results will only be as accurate as your diagram. In the best case, using a sharp pencil and carefully measuring lengths and angles, you can only expect about two significant digits of accuracy from a hand-drawn triangle. Nevertheless, even a roughly drawn triangle can give you an idea of the correct answers and be used to check your work after you use another method to solve the problem.

3.4.4 Trigonometric Method

The general approach for solving particle equilibrium problems using the trigonometric method is to:

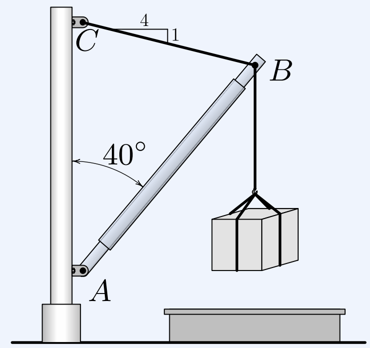
1. Draw and label a free-body diagram.
2. Rearrange the forces into a force triangle and label it.
3. Identify the knowns and unknowns.
4. Use trigonometry to find the unknown sides or angles of the triangle.

There must be no more than two unknowns to use this method, which may be either magnitudes or directions. During the problem setup, you will probably need to use the geometry of the situation to find one or more angles.

If the force triangle has a right angle you can use Section **??** to find the unknown values, but in most cases, the triangle will be oblique and you will need to use either or both of the Law of Sines or the Law of Cosines to find the sides or angles.

Example 3.4.4 Cargo Boom.

A 24 kN crate is being lowered into the cargo hold of a ship. Boom AB is 20 m long and acts at a 40° angle from kingpost AC. The boom is held in this position by topping lift BC which has a 1:4 slope.
Determine the forces in the boom and in the topping lift.

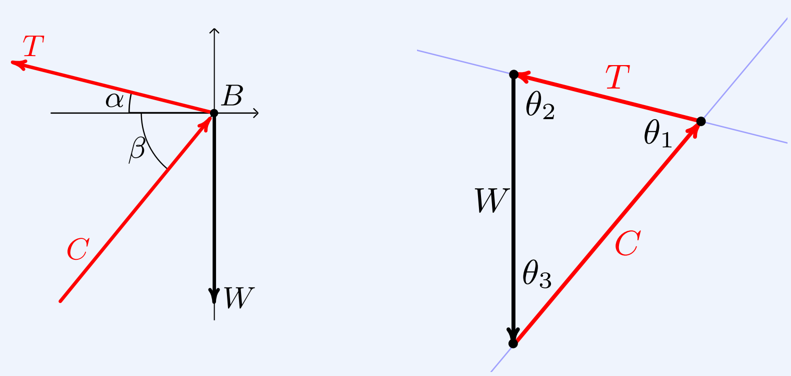


Solution.

1. *Draw diagrams.*

   Start by identifying the particle and drawing a free-body diagram. The particle in this case is point B at the end of the boom because it is the point where all three forces intersect. Let T be the tension of the topping lift, C be the force in the boom, and W be the weight of the load. Let α and β be the angles that forces T and C make with the horizontal.

   Rearrange the forces acting on point B to form a force triangle as was done in the previous example.



2. *Find angles.*

   Angle α can be found from the slope of the topping lift.

$$\alpha = \tan^{-1}\left(\frac{1}{4}\right) = 14.0°.$$

Angle β is the complement of the 40° angle the boom makes with the vertical kingpost.

$$\beta = 90° - 40° = 50°$$

Use these values to find the three angles in the force triangle.

$$\begin{aligned}\theta_1 &= \alpha + \beta = 64.0° \\ \theta_2 &= 90° - \alpha = 76.0° \\ \theta_3 &= 90° - \beta = 40.0°\end{aligned}$$

3. *Solve force triangle.*

   With the angles and one side of the force triangle known, apply the Law of Sines to find the two unknown sides.

$$\frac{\sin\theta_1}{W} = \frac{\sin\theta_2}{C} = \frac{\sin\theta_3}{T}$$

$$\begin{aligned}T &= W\left(\frac{\sin\theta_3}{\sin\theta_1}\right) & C &= W\left(\frac{\sin\theta_2}{\sin\theta_1}\right) \\ T &= 24 \text{ kN}\left(\frac{\sin 40.0°}{\sin 64.0°}\right) & C &= 24 \text{ kN}\left(\frac{\sin 76.0°}{\sin 64.0°}\right) \\ T &= 17.16 \text{ kN} & C &= 25.9 \text{ kN}\end{aligned}$$

3.4.5 Scalar Components Method

The general statement of equilibrium of forces, (3.1.1), can be expressed as the sum of forces in the **i**, **j** and **k** directions

$$\Sigma\mathbf{F} = \Sigma F_x\ \mathbf{i} + \Sigma F_y\ \mathbf{j} + \Sigma F_z\ \mathbf{k} = 0. \tag{3.4.1}$$

This statement will only be true if all three coefficients of the unit vectors are themselves equal to zero, leading to this scalar interpretation of the equilibrium equation

$$\Sigma\mathbf{F} = 0 \implies \begin{cases}\Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma F_z = 0\end{cases} \qquad \text{(three dimensions)}. \tag{3.4.2}$$

In other words, the single vector equilibrium equation is equivalent to three independent scalar equations, one for each coordinate direction.

In two-dimensional situations, no forces act in the **k** direction leaving just these two equilibrium equations to be satisfied

$$\Sigma \mathbf{F} = 0 \implies \begin{cases} \Sigma F_x = 0 \\ \Sigma F_y = 0 \end{cases} \qquad \text{(two-dimensions).} \tag{3.4.3}$$

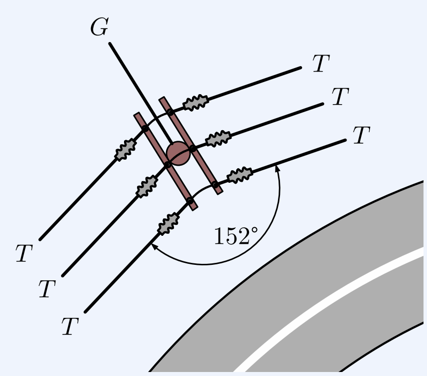
We will use this equation as the basis for solving two-dimensional particle equilibrium problems in this section and equation (3.4.2) for three-dimensional problems in Section 3.5.

You are undoubtedly familiar with utility poles, which carry electric, cable and telephone lines, but have you ever noticed as you drive down a winding road that the poles will switch from one side of the road to the other and back again? Why is this?

If you consider the forces acting on the top of a pole beside a curving section of road you'll observe that the tensions of the cables produce a net force towards the road. This force is typically opposed by a "guy wire" pulling in the opposite direction which prevents the pole from tipping over due to unbalanced forces. The power company tries to keep poles beside road segments with convex curvature. If they didn't switch sides, the guy wire for poles at concave curves would extend into the road... which is a poor design.

Example 3.4.5 Utility Pole.

Consider the utility pole next to the road shown below. A top view is shown in the right-hand diagram. If each of the six power lines pulls with a force of 10.0 kN, determine the magnitude of the tension in the guy wire.



Solution.

1. *Assumptions*.

A utility pole isn't two-dimensional, but we can solve this problem as if it was by first considering the force components acting in a horizontal plane, and then considering the components in a vertical plane.

It also isn't a concurrent force problem because the lines of action of the forces don't all intersect at a single point. However, we can make it into one by replacing the forces of the three power lines in each direction with a single force three times larger. This is an example of an *equivalent transformation*, a trick engineers use frequently to turn complex situations into simpler ones. It works here because all the tensions are equal, and the outside wires are equidistant from the center wire. You must be careful to justify all equivalent transformations because they will lead to errors if they are not applied correctly. Equivalent transformations will be discussed in greater detail in Section 4.7 later.

2. *Givens.*

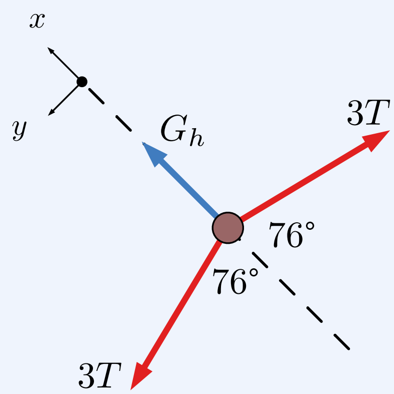
   $T = 10.0$ kN and 38° and 152° angles.

3. *Free-Body Diagram.*

   Begin by drawing a neat, labeled, free-body diagram of the top view of the pole, establishing a coordinate system and indicating the directions of the forces.

   Call the tension in one power line T and the tension in the guy wire G. Resolve the the tension of the guy wire into a horizontal component G_h, and a vertical component G_v. Only the horizontal component of G is visible in the top view.

   Although it is not necessary, it simplifies this problem considerably to note the symmetry and establish the x axis along the axis of symmetry.

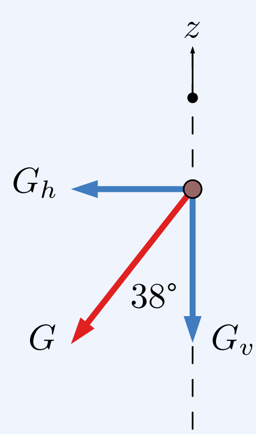


4. *Solution.*

   Solve for G_h by applying the equations of equilibrium. The symmetry of this problem means that the ΣF_x equation is sufficient.

$$\begin{aligned}\Sigma F_x &= 0\\ G_h - 6\,T_x &= 0\\ G_h &= 6\,(T\cos 76°)\\ &= 14.5\text{ kN}\end{aligned}$$

   Once G_h is determined, the tension of the guy wire G is easily found by considering the components of G in the side view. Note that the vertical component G_v tends to compress the pole.



$$G_h/G = \sin 38°$$

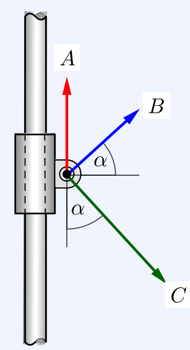
$$G = G_h / \sin 38°$$
$$G = 23.6 \text{ kN}$$

This problem could have also been solved using the force triangle method. See Subsection 3.4.3.

In the next example we look at the conditions of equilibrium by considering the load and the constraints, rather than taking a global equilibrium approach which considers both the load and reaction forces.

Example 3.4.6 Slider.

Three forces act on a machine part that is free to slide along a vertical, frictionless rod. Forces A and B have a magnitude of 20 N and force C has a magnitude of 30 N. Force B acts α degrees from the horizontal, and force C acts at the same angle from the vertical.
Determine the angle α required for equilibrium, and the magnitude and direction of the reaction force acting on the slider.

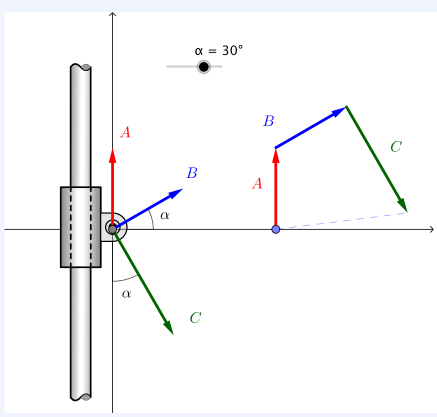


Solution.

1. *Givens.*

   We are given magnitudes of forces $A = 20$ N, $B = 20$ N, and $C = 30$ N. The unknowns are angle α and resultant force R.

2. *Procedure.*



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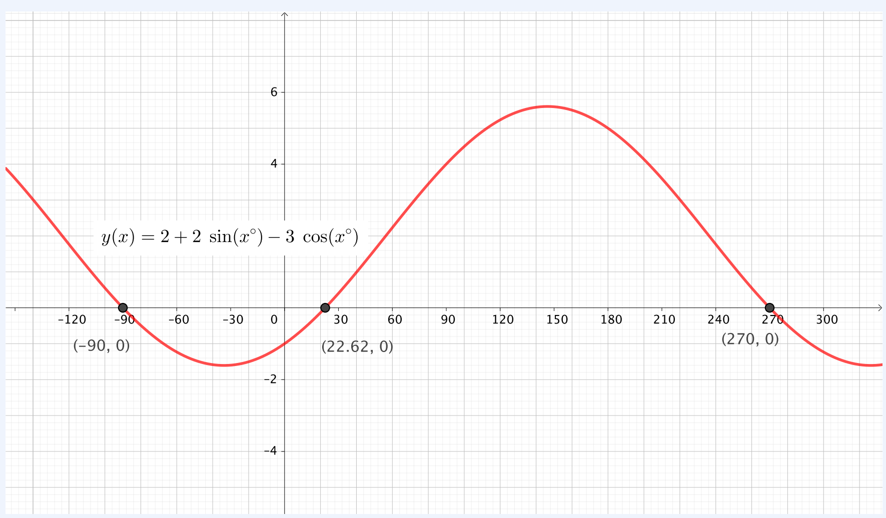
Since the rod is frictionless, it cannot prevent the slider from moving vertically. Consequently, the slider will only be in equilibrium if the resultant of the three load forces is horizontal. Since a horizontal force has no y component, we can establish this equilibrium condition:

$$R_y = \Sigma F_y = A_y + B_y + C_y = 0$$

Inserting the known values into the equilibrium relation and simplifying gives an equation in terms of unknown angle α.

$$\begin{aligned} R_y = A_y + B_y + C_y &= 0 \\ A + B\sin\alpha - C\cos\alpha &= 0 \\ 20 + 20\sin\alpha - 30\cos\alpha &= 0 \\ 2 + 2\sin\alpha - 3\cos\alpha &= 0 \end{aligned}$$

This is a single equation with a single unknown, although it is not particularly easy to solve with algebra. One approach is described at socratic.org[2]. An alternate approach is to use technology to graph the function $y(x) = 2 + 2\sin x - 3\cos x$. The roots of this equation correspond to values of α which satisfy the equilibrium condition above. The root occurring closest to $x = 0$ will be the answer corresponding to our problem, in this case $\alpha = 22.62°$ which you can verify by plugging it back into the equilibrium equation. Note that -90° also satisfies this equation, but it is not the solution we are looking for.



Once α is known, we can find the reaction force by adding the x components of A, B, and C.

$$R_x = A_x + B_x + C_x$$

$$
\begin{aligned}
&= A + B\cos\alpha + C\sin\alpha \\
&= 0 + 20\cos(22.62°) + 30\sin(22.62°) \\
&= 30.00 \text{ N}
\end{aligned}
$$

The resultant force $\mathbf{R}$ is the vector sum of R_x and R_y, but in this situation R_y is zero, so the resultant acts purely to the right with a magnitude of R_x.

$$\mathbf{R} = 30.00 \text{ N} \rightarrow .$$

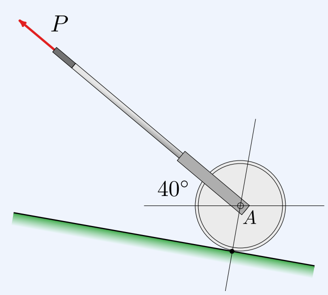
Note that this value is the **resultant** force, i.e. the net force applied to the slider by **A**, **B**, and **C**. However, the question asks for the **reaction** force, which is the force required for equilibrium. The reaction is equal and opposite to the resultant.

$$\mathbf{R}' = -\mathbf{R} = 30.00 \text{ N} \leftarrow$$

The next example demonstrates how rotating the coordinate system can simplify the solution. In the first solution, the standard orientation of the x and y axes is chosen, and in the second the coordinate system is rotated to align with one of the unknowns, which enables the solution to be found without solving simultaneous equations.

Example 3.4.7 Roller.

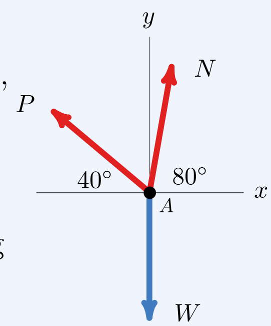
A lawn roller which weighs 160 lb is being pulled up a 10° slope at a constant velocity. Determine the required pulling force P.



Solution 1.

1. *Strategy.*

[2] socratic.org/questions/59e5f259b72cff6c4402a6a5



(a) Select a coordinate system, in this case, horizontal and vertical.

(b) Draw a free-body diagram

(c) Solve the equations of equilibrium using the scalar approach.

2. *Procedure.*

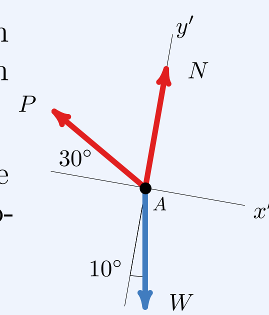
$$\begin{aligned} \Sigma F_x &= 0 \\ -P_x + N_x &= 0 \\ N \cos 80^\circ &= P \cos 40^\circ \\ N &= P\left(\frac{0.766}{0.174}\right) \end{aligned} \qquad \begin{aligned} \Sigma F_y &= 0 \\ P_y + N_y &= 0 \\ P \sin 40^\circ + N \sin 80^\circ &= W \\ 0.643P + 0.985N &= 160 \text{ lb} \end{aligned}$$

Solving simultaneously for P

$$\begin{aligned} 0.643P + 0.985(4.40P) &= 160 \text{ lb} \\ 4.98P &= 160 \text{ lb} \\ P &= 32.1 \text{ lb} \end{aligned}$$

Solution 2.

1. *Strategy.*



(a) Rotate the standard coordinate system 10° clockwise to align the new y' axis with force N.

(b) Draw a free-body diagram and calculate the angles between the forces and the rotated coordinate system.

(c) Solve for force P directly.

2. *Procedure.*

$$\Sigma F_{x'} = 0$$

$$
\begin{aligned}
-P_{x'} + W_{x'} &= 0 \\
P \cos 30^\circ &= W \sin 10^\circ \\
P &= 160 \text{ lb} \left(\frac{0.1736}{0.866}\right) \\
P &= 32.1 \text{ lb}
\end{aligned}
$$

3.4.6 Multi-Particle Equilibrium

When two or more particles interact with each other there will always be common forces between them as a result of Newton's Third Law, the action-reaction principle.

Consider the two boxes with weights W_1 and W_2 connected to each other and the ceiling shown in the interactive diagram. Position one shows the physical arrangement of the objects, position two shows their free-body diagrams, and position three shows simplified free-body diagrams where the objects are represented by points. The boxes were freed by replacing the cables with tension forces T_A and T_B.

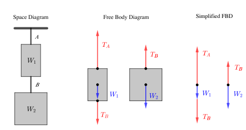


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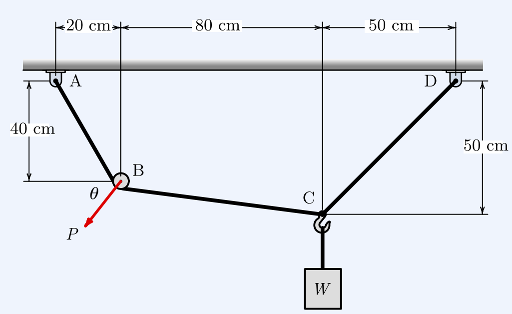
Figure 3.4.8 Two suspended weights

From the free-body diagrams you can see that cable B only supports the weight of the bottom box, while cable A and the ceiling support the combined weight. The tension T_B is common to both diagrams. Recognizing the common force is the key to solving multi-particle equilibrium problems.

Example 3.4.9 Two hanging weights.

A 100 N weight W is supported by cable $ABCD$. There is a frictionless pulley at B and the hook is firmly attached to the cable at point C.
What is the magnitude and direction of force **P** required to hold the system in the position shown?

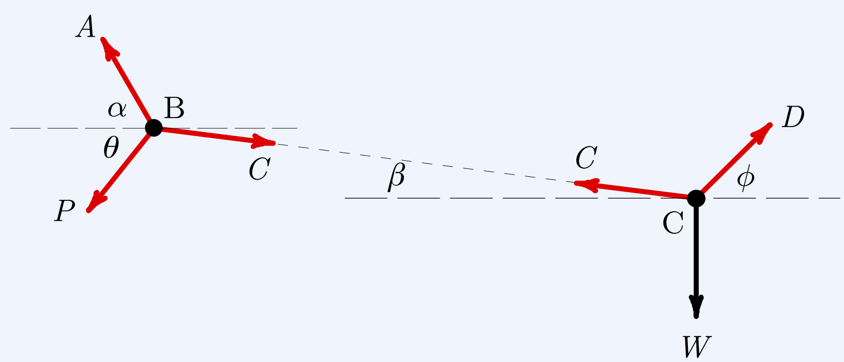


Hint. The particles are points B and C. The common force is the tension in rope segment BC.

Solution.

1. *Strategy.*

   Following the General Procedure we identify the particles as points A and B, and draw free-body diagrams of each. We label the rope tensions A, C, and D for the endpoints of the rope segments, and label the angles of the forces α, β, and ϕ. We will use the standard Cartesian coordinate system and use the scalar components method.



   Weight W was given, and we can easily find angles α, β, and ϕ so the knowns are:

$$\begin{aligned} W &= 100 \text{ N} \\ \alpha &= \tan^{-1}\left(\frac{40}{20}\right) = 63.4° \\ \beta &= \tan^{-1}\left(\frac{10}{80}\right) = 7.13° \\ \phi &= \tan^{-1}\left(\frac{50}{50}\right) = 45° \end{aligned}$$

   Counting unknowns we find that there are two on the free-body diagram of particle C (C and D), but four on particle B, (A C, P and θ).

   Two unknowns on particle C means it is solvable since there are two equilibrium equations available, so we begin there.

2. *Solve Particle C.*

$$\begin{aligned} \Sigma F_x &= 0 \\ -C_x + D_x &= 0 \\ C\cos\beta &= D\cos\phi \end{aligned} \qquad \begin{aligned} \Sigma F_y &= 0 \\ C_y + D_y - W &= 0 \\ C\sin\beta + D\sin\phi &= W \end{aligned}$$

$$C = D\left(\frac{\cos 45°}{\cos 7.13°}\right) \qquad C\sin 7.13° + D\sin 45° = 100 \text{ N}$$
$$C = 0.713D \qquad 0.124C + 0.707D = 100 \text{ N}$$

Solving these two equations simultaneously gives

$$C = 89.6 \text{ N} \qquad D = 125.7 \text{ N}.$$

With particle C solved, we can use the results to solve particle B. There are three unknowns remaining, tension A, magnitude P, and direction θ. Unfortunately, we still only have two available equilibrium equations. When you find yourself in this situation with more unknowns than equations, it generally means that you are missing something. In this case, it is the pulley. When a cable wraps around a frictionless pulley the tension doesn't change. The missing information is that $A = C$. Knowing this, the magnitude and direction of force $\mathbf{P}$ can be determined.

Because $A = C$, the free-body diagram of particle B is symmetric, and the technique used in Example 3.4.5 to rotate the coordinate system could be applied here.

3. *Solve Particle B.*

   Referring to the FBD for particle B we can write these equations.

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0$$
$$-A_x - P_x + C_x = 0 \qquad A_y - P_y - C_y = 0$$
$$P\cos\theta = C\cos\beta - A\cos\alpha \qquad P\sin\theta = A\sin\alpha - C\sin\beta$$

   Since $A = C = 89.6$ N, substituting and solving simultaneously gives

$$P\cos\theta = 48.8 \text{ N} \qquad P\sin\theta = 69.0 \text{ N}$$
$$P = 84.5 \text{ N} \qquad \theta = 54.7°.$$

   These are the magnitude and direction of vector $\mathbf{P}$. If you wish, you can express $\mathbf{P}$ in terms of its scalar components. The negative signs on the components have been applied by hand since $\mathbf{P}$ points down and to the left.

$$\begin{aligned}\mathbf{P} &= \langle -P\cos\theta, -P\sin\theta\rangle \\ &= \langle -48.8 \text{ N}, -69.0 \text{ N}\rangle\end{aligned}$$

3.5 3D Particle Equilibrium

The world we live in has three dimensions. One and two-dimensional textbook problems have been useful for learning the principles of engineering mechanics, but to model real-world problems we will have to consider all three.

Fortunately, all the principles you have learned so far still apply, but many students have difficulty visualizing three-dimensional problems drawn on two-dimensional paper and the mathematics becomes a bit harder. It is especially important to have good diagrams and keep your work neat and organized to avoid errors.

3.5.1 Three-Dimensional Coordinate Frame

We need a coordinate frame for three dimensions, just as we did in two dimensions, so we add a third orthogonal axis z to our existing two-dimensional frame.

For equilibrium of a particle, usually the origin of the coordinate frame is at the particle, the x axis is horizontal, and the y axis is vertical just as in a two-dimensional situation. The orientation of the z axis is determined by the **right-hand rule**. Using your *right hand*, put your palm at the origin and point your fingers along the positive x axis. Then curl your fingers towards the positive y axis. Your thumb will point in the direction of the positive z axis. For example, in the plane of the page with the positive x axis horizontal and to the right and the positive y axis vertical and upwards, the positive z axis will point *towards* you out of the page. Remember that the three axes are mutually perpendicular, i.e. each axis is perpendicular to both of the others. The right-hand rule is important in many aspects of engineering, so make sure that you understand how it works. Mistakes will lead to sign errors.

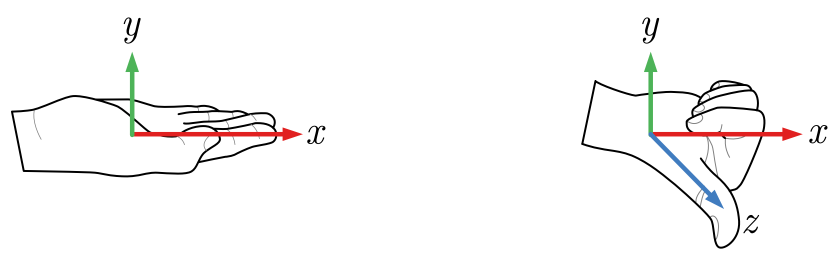


Figure 3.5.1 Point-and-curl right-hand rule technique.

3.5.2 Free-Body Diagrams

As we did before, we begin our analysis by drawing a free-body diagram that shows all forces and moments acting on the object of interest. Drawing a FBD in three dimensions can be difficult. It is sometimes hard to see things in three dimensions when they are drawn on a two-dimensional sheet. Consequently, it is important to carefully label vectors and angles, but not to clutter up the diagram with too much and/or unnecessary information. When working in two dimensions, you only need one angle to determine the direction of the vector, but when working in three dimensions you need two or three angles.

3.5.3 Angles

As stated above, when working in three dimensions you need three angles to determine the direction of the vector, namely, the angle with respect to the x axis, the angle with respect to the y axis and the angle with respect to the z axis. The three angles mentioned above are not necessarily located in any of the coordinate planes. Think of it this way — three points determine a plane, and in this case, the three points are: the origin, the tip of the vector, and a point on an axis. The plane made by those three points is not necessarily the xy, yz, or xz plane. It is most likely a "tilted" plane.

As you learned in Subsection 2.4.2, one way to quantify the direction of a vector is with direction cosine angles. These direction cosine angles are measured from the positive x, y, and z axes and are often labeled $_x$, $_y$, and $_z$, respectively.

As with two dimensions, angles can be determined from geometry — a distance vector going in the same direction as the force vector. This is the three-dimensional equivalent of similar triangles that you used in the two-dimensional problems.

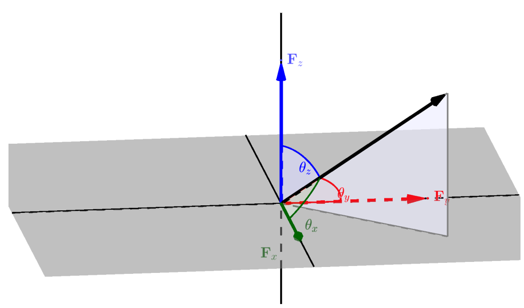


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Figure 3.5.2 Direction Cosine Angles

If you know that the line of action of a force vector goes between two points, then you can use the distance vector that goes from one point to the other to determine the angles.

Let's suppose that the line of action goes through two points A and B, and the direction of the force is from A towards B. The first step in determining the three angles is to write the distance vector from point A towards point B. Let's call this vector $\mathbf{r}_{AB}$. Starting at point A, you need to determine how to get to point B by moving in each of the three directions. Ask yourself: to get from point A to point B do I have to move in the x direction? If so, how far do I have to travel? This becomes the x component of the vector $\mathbf{r}_{AB}$ namely r_{AB_x}. Next, to get from point A to point B how far do I move in y direction? This distance is r_{AB_y}. Finally, to get from point A to point B how far do I move in the z-direction? This distance is r_{AB_z}.

When writing these scalar components pay attention to which way you move along the axes. If you travel toward the positive end of an axis, the corresponding scalar component gets a positive sign. Travel towards the negative end results in a negative sign. The sign is important.

Once you have determined the components of the distance vector r_{AB}, you can determine the total distance from point A to B using the three-dimensional Pythagorean Theorem

$$r_{AB} = \sqrt{(r_{AB_x})^2 + (r_{AB_y})^2 + (r_{AB_z})^2} \tag{3.5.1}$$

Lastly, the angles are determined by the direction cosines, namely

$$\cos\theta_x = \frac{r_{AB_x}}{r_{AB}} \qquad \cos\theta_y = \frac{r_{AB_y}}{r_{AB}} \qquad \cos\theta_z = \frac{r_{AB_z}}{r_{AB}}$$

Since the force vector has the same line of action as the distance vector, by the three-dimensional version of similar triangles,

$$\frac{r_{AB_x}}{r_{AB}} = \frac{F_x}{F} \qquad \frac{r_{AB_y}}{r_{AB}} = \frac{F_y}{F} \qquad \frac{r_{AB_z}}{r_{AB}} = \frac{F_z}{F}.$$

So,

$$F_x = \left(\frac{r_{AB_x}}{r_{AB}}\right) F \qquad F_y = \left(\frac{r_{AB_y}}{r_{AB}}\right) F \qquad F_z = \left(\frac{r_{AB_z}}{r_{AB}}\right) F$$

Now, that is a bit of math there, but the important things to remember are:

- You can use three angles to determine the direction of a force in three dimensions.
- You can use the geometry to get them from a distance vector that lies along the line of action of the force.

The three direction cosine angles are not mutually independent. From (3.5.1) you can easily show that

$$\cos\theta_x^2 + \cos\theta_y^2 + \cos\theta_z^2 = 1, \tag{3.5.2}$$

so if you know two direction cosine angles you can find the third from this relationship.

3.5.4 General Procedure

The general procedure for solving three-dimensional particle equilibrium is essentially the same as for two-dimensional particle equilibrium using the components method. The major differences are that you must carefully find each vector component using the techniques from Section 2.4. The process follows the same five-step method for creating a free-body diagram, followed by steps to solve your equilibrium equations.

Draw a Free-body Diagram:

1. *Select and isolate the particle.* The “free-body” in free-body diagram means that a concurrent force particle or connection must be isolated from the supports that are physically holding it in place. This means creating a separate free-body diagram from your problem sketch.

2. *Establish a coordinate system.* Draw a right-handed coordinate system to use as a reference for your equilibrium equations. Look ahead and select a coordinate system that minimizes the number of force components. This will simplify your vector algebra. The choice is technically arbitrary, but a good choice will simplify your calculations and reduce your effort.

3. *Identify all loads.* Add force vectors to your free-body diagram representing each applied load pushing or pulling the body, in addition to the body’s weight, if it is non-negligible. If a force vector has a known direction, draw it. If its direction is unknown, assume one, and your later algebra will check your assumption. Every vector should have a descriptive variable name and a clear arrowhead indicating its direction.

4. *Identify all reactions.* Reactions represent the resistance of the physical supports you cut away by isolating the body in step 1. All particle supports are some type of two-force members with tension or compression reaction forces. These reactions will all be concurrent with the body loads from Step 2. Label each reaction with a descriptive variable name and a clear arrowhead. Again, if a vector’s direction is unknown, just assume one.

5. *Label the diagram.* Verify that every dimension, angle, force, and moment is labeled with either a value or a symbolic name if the value is unknown. In our eyes, dimensioning is optional. Having the information needed for your calculations is helpful, but don’t clutter the diagram up with unneeded details. Your final free-body diagram should be a stand-alone presentation and is the basis of your equilibrium equations.

Create and Solve Equilibrium Equations

1. *Break vectors into components.* Compute each force’s x, y, and z components using the tools outlined in Section 2.4. While the components in two-dimensional problems can often be found with right triangle trigonometry, three-dimensional problems often use unit vectors.

2. *Write equilibrium equations.* Now represent your free-body diagram as equilibrium equations. For a three-dimensional particle equilibrium problem, you can have up to three force equilibrium equations corresponding to a force balance in the three independent x, y, and z directions. Each equation should start with the governing equation, like $\Sigma F_x = 0$.

3. *Count knowns and unknowns.* At this point, you should have at most three unknowns remaining. If you have over three, reread the problem and look for overlooked information.

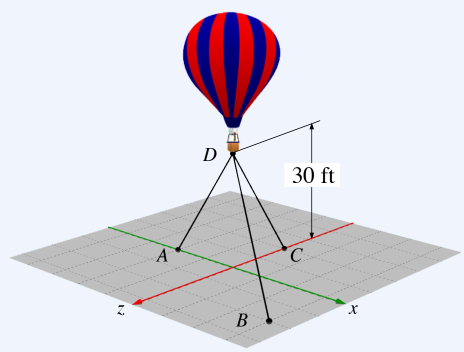
4. *Solve for unknowns.* Use algebra to simplify the equilibrium equations and solve for unknowns. With multiple unknowns scattered across multiple equations, linear algebra may be more efficient than substitution. Assume that all answers have units - unless you prove that they don't. Finally, underline or box your answers.

5. *Check your work.* If you add the components of the forces, do they add to zero? Do the results seem reasonable given the situation? Have you included appropriate units?

Now let's see how that process looks on an example problem.

Example 3.5.3 Balloon.

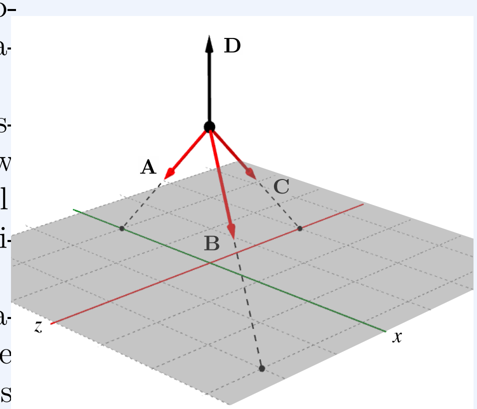
A hot air balloon 30 ft above the ground is tethered by three cables as shown in the diagram.
If the balloon is pulling upwards with a force of 900 lb, what is the tension in each of the three cables?
The grid lines on the ground plane are spaced 10 ft apart.



Solution.

1. *Strategy.*

   The three tensions are the unknowns which we can find by applying the three equilibrium equations.
   We'll establish a coordinate system with the origin directly below the balloon and the y axis vertical then draw and label a free-body diagram.
   Next we'll use the given information to find two points on each line of action to find the components of each force in terms of the unknowns.

   When the x, y and z components of all forces can be expressed in terms of known values, the equilibrium equations can be solved.

2. *Geometry.*

From the diagram, the coordinates of the points are

$$\text{A} = (-20, 0, 0) \quad \text{B} = (30, 0, 20) \quad \text{C} = (0, 0, -20) \quad \text{D} = (0, 30, 0)$$

Use the point coordinates to find the x, y and z components of the forces.

$$\begin{aligned}
A_x &= \frac{-20}{L_A}A & A_y &= \frac{-30}{L_A}A & A_z &= \frac{0}{L_A}A \\
B_x &= \frac{30}{L_B}B & B_y &= \frac{-30}{L_B}B & B_z &= \frac{20}{L_B}B \\
C_x &= \frac{0}{L_C}C & C_y &= \frac{-30}{L_C}C & C_z &= \frac{-20}{L_C}C
\end{aligned}$$

Where L_A, L_B and L_C are the lengths of the three cables found with the distance formula.

$$\begin{aligned}
L_A &= \sqrt{(-20)^2 + (-30)^2 + 0^2} &&= 36.1 \text{ ft} \\
L_B &= \sqrt{30^2 + (-30)^2 + 20^2} &&= 46.9 \text{ ft} \\
L_C &= \sqrt{0^2 + (-30)^2 + (-20)^2} &&= 36.1 \text{ ft}
\end{aligned}$$

3. *Equilibrium Equations.*

   Applying the three equations of equilibrium yields three equations in terms of the three unknown tensions.

$$\begin{aligned}
\Sigma F_x &= 0 \\
&A_x + B_x + C_x = 0 \\
&-\frac{20}{36.1}A + \frac{30}{46.9}B + 0\,C = 0 \\
A &= 1.153\,B \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\Sigma F_z &= 0 \\
&A_z + B_z + C_z = 0 \\
&0\,A + \frac{20}{46.9}B - \frac{20}{36.1}C = 0 \\
C &= 0.769\,B \qquad (2)
\end{aligned}$$

$$\begin{aligned}
&\Sigma F_y = 0 \\
&\quad A_y + B_y + C_y + D = 0 \\
&\quad -\frac{30}{36.1}A - \frac{30}{46.9}B - \frac{30}{36.1}C + 900 = 0 \\
&\quad 0.832\,A + 0.640\,B + 0.832\,C = 900 \text{ lb} \qquad (3)
\end{aligned}$$

Solving these equations simultaneously yields the answers we are seeking. One way to do this is to substitute equations (1) and (2) into (3) to eliminate A and C and solve the resulting equation for B.

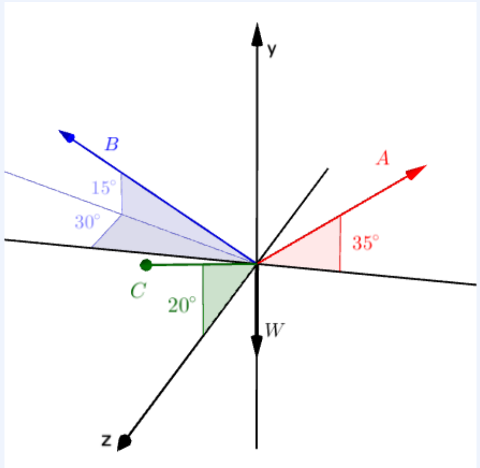
$$\begin{aligned}
0.832\,(1.153\,B) + 0.640\,B + 0.832\,(0.769\,B) &= 900 \text{ lb} \\
2.24B &= 900 \text{ lb} \\
B &= 402 \text{ lb}
\end{aligned}$$

With B known, substitute it into equations (1) and (2) to find A and C.

$$\begin{aligned}
A &= 1.153\,B & C &= 0.769\,B \\
&= 464 \text{ lb} & &= 309 \text{ lb}
\end{aligned}$$

Example 3.5.4 Skycam.

The skycam at Stanford University Stadium has a mass of 20 kg and is supported by three cables as shown. Assuming that it is currently in equilibrium, find the tension in each of the three supporting cables.



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Solution. In this situation, the directions of all four forces are specified by the angles in the free-body diagram, and the magnitude of the weight is known. The three unknowns are the magnitudes of forces **A**, **B**, and **C**.

$$W = mg = 20 \text{ kg } 9.81 \text{ m/s}^2 = 196.2 \text{ N}$$

We will first find unit vectors in the directions of the four forces by inspection of the free-body diagram. This step requires visualizing the component's unit vectors and determining the angles each makes with the coordinate axis.

$$\begin{aligned}
\hat{\mathbf{W}} &= \langle 0, -1, 0 \rangle \\
\hat{\mathbf{A}} &= \langle \cos 35°, \cos 55, 0 \rangle \\
\hat{\mathbf{B}} &= \langle -\cos 15° \cos 30°, \cos 75°, -\cos 15° \cos 60° \rangle \\
\hat{\mathbf{C}} &= \langle 0, \cos 70, \cos 20° \rangle
\end{aligned}$$

Particle equilibrium requires that $\sum \mathbf{F} = 0$.

$$A\ \hat{\mathbf{A}} + B\ \hat{\mathbf{B}} + C\ \hat{\mathbf{C}} = -W\ \hat{\mathbf{W}}$$

This is a 3×3 system of three simultaneous equations, one for each coordinate direction, which needs to be solved for A, B, and C.

$$\begin{aligned}
A \cos 35° - B \cos 15° \cos 30° + 0 &= 0 && (\Sigma F_x = 0) \\
A \cos 55° + B \cos 75° + C \cos 70° &= 196.2 \text{ N} && (\Sigma F_y = 0) \\
0 - \cos 15° \cos 60° + C \cos 20° &= 0 && (\Sigma F_z = 0)
\end{aligned}$$

These can be solved by any method you choose. Here we will use Sage. Evaluating the coefficients and expressing the equations in matrix form gives

$$\begin{bmatrix} 0.819 & -0.837 & 0 \\ 0.574 & 0.259 & 0.342 \\ 0 & -0.482 & 0.940 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 196.2 \text{ N} \\ 0 \end{bmatrix}.$$

This is an equation in the form

$$[A][x] = [B].$$

Entering the coefficient matrices into Sage.

```
A = Matrix([[0.819,-0.837,0],[0.574,0.259,
    0.342],[0,-0.482,0.940]])
B = vector([0, 196.2, 0])
x = A.solve_right(B)
x
```

```
(196.391530042156, 192.168056277808, 98.5372373679827)
```

After evaluating, we learn that

$$A = 196.4 \text{ N} \qquad B = 192.2 \text{ N} \qquad C = 98.5 \text{ N}.$$

3.6 Exercises (Ch. 3)

Particle Equilibrium	**0/170**
Equilibrium equation method	0/20 Not attempted
Four forces	0/20 Not attempted
Hanging weight	0/30 Not attempted
Ball in a trough	0/20 Not attempted
Rope and pulley	0/20 Not attempted
3D Equilibrium: Hanging Plate	0/60 Not attempted

Multi-particle Equilibrium	**0/50**
Two suspended loads	0/20 Not attempted
Cylinders in a trough	0/30 Not attempted

Created using Numbas, developed by Newcastle University.

Chapter 4

Moments and Static Equivalence

When a force is applied to a body, the body tends to translate in the direction of the force and also tends to rotate. We have already explored the translational tendency in Chapter 3. We will focus on the rotational tendency in this chapter.

This rotational tendency is known as the moment of the force, or more simply the **moment**. You may be familiar with the term *torque* from physics. Engineers generally use "moment" whereas physicists use "torque" to describe this concept. Engineers reserve "torque" for moments that are applied about the long axis of a shaft and produce **torsion**.

Moments are vectors, so they have magnitude and direction and obey all rules of vector addition and subtraction described in Chapter 2. Additionally, moments have a *center of rotation*, although it is more accurate to say that they have an axis of rotation. In two dimensions, the axis of rotation is perpendicular to the plane of the page and so will appear as a **point of rotation**, also called the **moment center**. In three dimensions, the axis of rotation can be any direction in 3D space.

A wrench provides a familiar example. A force $\mathbf{F}$ applied to the handle of a wrench, as shown in Figure 4.0.1, creates a moment $\mathbf{M}_A$ about an axis out of the page through the centerline of the nut at A. The $\mathbf{M}$ is bold because it represents a vector, and the subscript A indicates the axis or center of rotation. The direction of the moment can be either clockwise or counter-clockwise depending on how the force is applied.

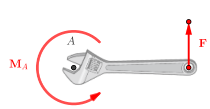

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Figure 4.0.1 A moment $\mathbf{M}_A$ is created about point A by force $\mathbf{F}$.

4.1 Direction of a Moment

In a two-dimensional problem the direction of a moment can be determined easily by inspection as either clockwise or counterclockwise. A counter-clockwise rotation corresponds with a moment vector pointing out of the page and is considered positive.

In three dimensions, a moment vector may point in any direction in space and is more difficult to visualize. The direction is established by the *right-hand rule*. Recall that in Subsection 2.8.1 that you were introduced to the right-hand rule and cross products.

To find a moment using the right-hand rule, first establish a **position vector** **r** pointing *from* the point of interest (the rotation center) *to* a point along the force's line-of-action. Next, there are two options for physically finding the direction of the moment from the right-hand rule, the three-finger or slide-and-curl methods.

To use the three-finger method, align your right-hand index finger with the position vector and your middle finger with the force vector, then your thumb will point in the direction of the moment vector. Alternately, if you align your thumb with the position vector and your index finger with the force vector, then your middle finger points in the direction of the moment vector **M**

Figure 4.1.1 Three finger right-hand rule techniques for moments.

Another approach is the point-and-curl method. Start with your right hand flat and fingertips pointing along the position vector **r** pointing *from* the center of rotation *to* a point on the force's line of action. Rotate your hand until the force **F** is perpendicular to your fingers and imagine that it pushes your fingers into a curl around your thumb. In this position, your thumb defines the axis of rotation, and points in the direction of the moment **M**.

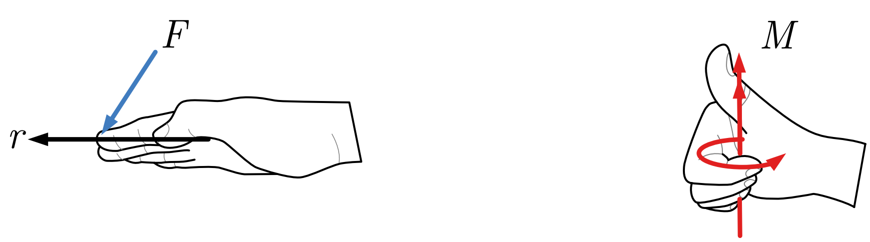

Figure 4.1.2 Point-and-curl right-hand rule technique for moments.

Consider the page shown below on a horizontal surface. Using these techniques, we see that a counter-clockwise moment vector points up, or out of the page, while the clockwise moment points down or into the page. In other words, the counter-clockwise moment acts in the positive z direction and the clockwise moment acts in the $-z$ direction.

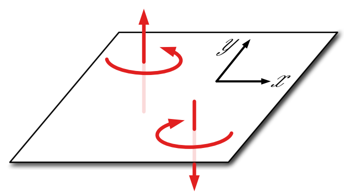

Figure 4.1.3 Moments in the plane of the page.

Any of these techniques may be used to find the direction of a moment. They all produce the same result so you don't need to learn them all, but make sure you have at least one method you can use accurately and consistently.

4.2 Magnitude of a Moment

Key Questions

- Why is there no moment about any point on the line of action of a force?
- If you increase the distance between a force and a point of interest, does the moment of the force go up or down?
- What practical applications can you think of that could use moments to describe?

As you probably know, the turning effect produced by a wrench depends on where and how much force you apply to the wrench, and the optimum direction to apply the force is at right angles to the wrench's handle. If the nut won't budge, you need to apply a larger force or get a longer wrench.

This strength of this turning effect is what we mean by the magnitude of a moment (or of a torque).

4.2.1 Definition of a Moment

The magnitude of a moment is found by multiplying the magnitude of force $\mathbf{F}$ times the **moment arm**, where the moment arm is defined as the **perpendicular distance,** $d_\perp$, from the center of rotation to the line of action of the force, measured perpendicularly as illustrated in the interactive.

$$M = F d_\perp. \tag{4.2.1}$$

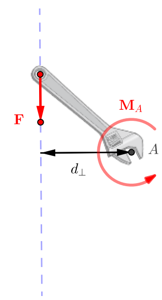

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Figure 4.2.1 Definition of the moment, $M = Fd_\perp$.

Notice that the magnitude of a moment depends only on the force and the moment arm, so the same force produces different moments about different points in space. The closer the center of rotation is to the force's line of action, the smaller the moment. Points on the force's line of action experience no moment because there the moment arm is zero. Furthermore, vector magnitudes are always positive, so clockwise and counter-clockwise moments with the same strength have the same magnitude.

4.3 Scalar Components

We saw in Subsection 3.3.2 that vectors can be expressed as the product of a **scalar component** and a unit vector.

For example, a 100 N force acting down can be represented by $F_y\,\mathbf{j}$, where F_y is the scalar component and $F_y = -100$ N. This describes a vector $\mathbf{F}$ which has a magnitude of 100 N and acts in the $-\mathbf{j}$ direction, i.e. $\downarrow$. The unit vector $\mathbf{j}$ along with the sign $(+/-)$ of F_y determines the direction, while the absolute value of F_y determines the vector's magnitude.

Moments in two dimensions are either clockwise or counter-clockwise, or alternately they point into or out of the page. This means that a single scalar value is sufficient to completely specify such a moment if we have established which direction is positive. The choice is arbitrary, but the default sign convention is based on the right-handed Cartesian coordinate system, as illustrated in Figure 4.1.3.

When using the standard convention, counter-clockwise moments are positive and clockwise moments are negative. Simply append a positive sign to the magnitude for counter-clockwise moments or a negative sign for clockwise moments to create a scalar component. You are free to use the opposite convention, but this should be explicitly stated.

Example 4.3.1 Sign Conventions.

For each scalar component, determine the direction of the corresponding moment vector.

A $M_1 = 30$ N·m

B $M_2 = -400$ kN·m

C $M_3 = 25$ N·m $\circlearrowright$

D $M_4 = -100$ ft·lb $\circlearrowright$

Solution.

A CCW. Use the default sign convention, i.e. CCW is positive.

B CW. Negative value means the moment acts opposite to positive direction.

C CW. The arrow overrides default sign convention, so now CW is positive direction.

D CCW. Negative CW is CCW.

Scalar components are most useful when combining several clockwise and counter-clockwise moments. The resulting algebraic sum of the scalar components will be either positive, negative, or zero, and this sign indicates the direction of the resultant moment.

Example 4.3.2 Scalar addition.

Use scalar moments to determine the magnitude of the resultant of three moments:
$\mathbf{M_1} = 25$ kN·m $\circlearrowright$, $\mathbf{M_2} = 40$ kN·m $\circlearrowleft$, and $\mathbf{M_3} = 30$ kN·m $\circlearrowright$

Solution. Manually attaching the signs according to the standard sign convention (CCW +) gives the scalar moments:

$$\begin{aligned} M_1 &= -25 \text{ kN·m} \\ M_2 &= +40 \text{ kN·m} \\ M_3 &= -30 \text{ kN·m}. \end{aligned}$$

Adding these moments gives the resultant scalar moment.

$$\begin{aligned} M &= M_1 + M_2 + M_3 \\ &= (-25 \text{ kN·m}) + (40 \text{ kN·m}) + (-30 \text{ kN·m}) \\ &= -15 \text{ kN·m}. \end{aligned}$$

The negative sign indicates that the resultant vector moment is clockwise. Interpreting the resultant as a vector gives:

$$\mathbf{M} = 15 \text{ kN·m} \circlearrowright .$$

The corresponding magnitude of $\mathbf{M}$ is

$$|\mathbf{M}| = 15 \text{ kN}\cdot\text{m}.$$

In three dimensions, moments, like forces, can be resolved into components in the x, y, and z directions.

$$\mathbf{M} = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_x\,\mathbf{k}.$$

This means that the three scalar components are required to fully specify a moment in three dimensions.

Warning 4.3.3

Be careful not to mix up magnitudes with scalar components.

- Both are scalar values with units.
- Magnitudes are never negative. Scalar components have a sign.
- Scalar components always have an associated sign convention. It may be implied or specifically indicated. By default counter-clockwise moments are positive.
- There is no special symbol or notation to indicate whether a quantity represents a vector magnitude or a scalar moment, so pay attention to context.

4.4 Varignon's Theorem

Varignon's Theorem is a method to calculate moments developed in 1687 by French mathematician Pierre Varignon (1654 – 1722). It states that sum of the moments of several concurrent forces about a point is equal to the moment of the resultant of those forces, or alternately, the moment of a force about a point equals the sum of the moments of its components.

This means you can find the moment of a force by first breaking it into components, evaluating the scalar moments of the individual components, and finally summing them to find the net moment about the point. The scalar moment of a component is the magnitude of the component times the perpendicular distance to the moment center by the definition of a moment, with a positive or negative sign assigned to indicate its direction.

This may sound like more work than just finding the moment of the original force, but in practice, it is often easier. Consider the interactive to the right. If we break the force into components along the wrench handle and perpendicular to it, the sum of the moments is

$$M = F_\perp d, \tag{4.4.1}$$

where d is the length of the handle, and $F_\perp$ is the component of F perpendicular to the handle. Here, the contribution of the parallel component to the sum is zero, since its line of action passes through the moment center A.

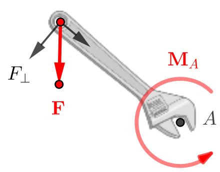

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Figure 4.4.1 Varignon's Theorem: $M = F_\perp d$

This result agrees with our intuitive understanding of how a wrench works; the greatest torque is developed when the force is applied at a right angle to the handle.

Equations (4.2.1) and (4.4.1) not only produce the same result, but they are also completely identical. If the length of the handle is d and the angle between the force $\mathbf{F}$ and the handle is θ, then $d_\perp = d \sin\theta$, and $F_\perp = F \sin\theta$. Using either equation to calculate the moment gives

$$M = F\ d\ \sin\theta. \tag{4.4.2}$$

4.4.1 Rectangular Components

Varignon's theorem is particularly convenient to use because the diagram provides horizontal and vertical dimensions, which is often the case. If you decompose forces into horizontal and vertical components you can find the scalar moments of the components without difficulty.

The moment of a force is the sum of the moments of the components.

$$M = \pm F_x d_y \pm F_y d_x \tag{4.4.3}$$

Take care to assign the correct sign to the individual moment terms to indicate direction; positive moments tend to rotate the object counter-clockwise and negative moments tend to rotate it clockwise according to the standard right-hand rule convention.

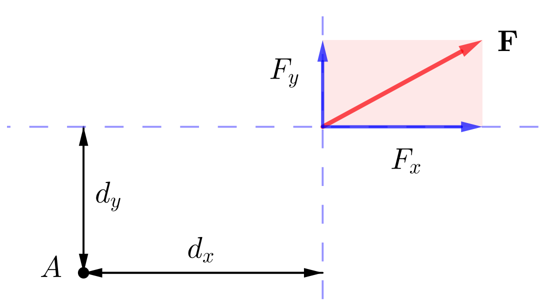
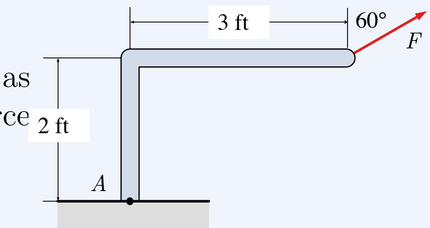


Figure 4.4.2 Sum of moments of components. $M = \pm F_x d_y \pm F_y d_x$

Example 4.4.3 Varignon's Theorem.

A 750 lb force is applied to the frame as shown. Determine the moment this force makes about point A.



Solution. Force **F** acts 60° from the vertical with a 750 lb magnitude, so its horizontal and vertical components are

$$F_x = F\sin 60° = 649.5 \text{ lb}$$
$$F_y = F\cos 60° = 375.0 \text{ lb}$$

For component F_x, the perpendicular distance from point A is 2 ft so the moment of this component is

$$M_1 = 2F_x = 1299 \text{ ft}\cdot\text{lb Clockwise.}$$

For component F_x, the perpendicular distance from point A is 3 ft so the moment of this component is

$$M_2 = 3F_y = 1125 \text{ ft}\cdot\text{lb Counter-clockwise.}$$

Assigning a negative sign to M_1 and a positive sign to M_2 to account for their directions and summing, gives the moment of **F** about A.

$$\begin{aligned} M_A &= -M_1 + M_2 \\ &= -1299 + 1125 \\ &= -174 \text{ ft}\cdot\text{lb} \end{aligned}$$

The negative sign indicates that the resultant moment is clockwise, with a magnitude of 174 ft·lb.

$$\mathbf{M}_A = 174 \text{ ft}\cdot\text{lb Clockwise.}$$

The interactive diagram below will help you visualize the different approaches for finding moments covered in this section.

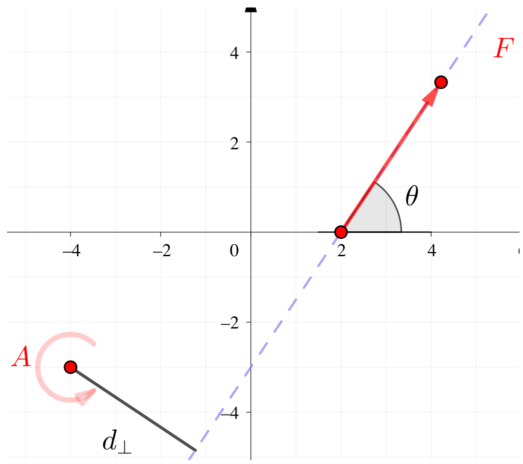


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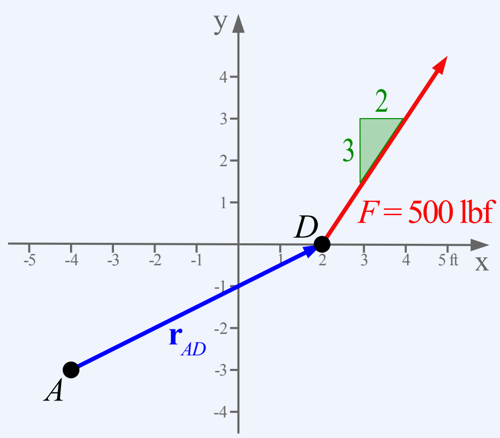
Figure 4.4.4 Three equivalent approaches to finding a moment about a point.

Example 4.4.5 2D Moments - Four Ways.



Force $\mathbf{F}$ has a magnitude of 500 lbf and acts on point D in the direction shown.

Find the moment caused by force $\mathbf{F}$ around point $A = (-4, -3)$ ft using different methods and verify that they give the same result.

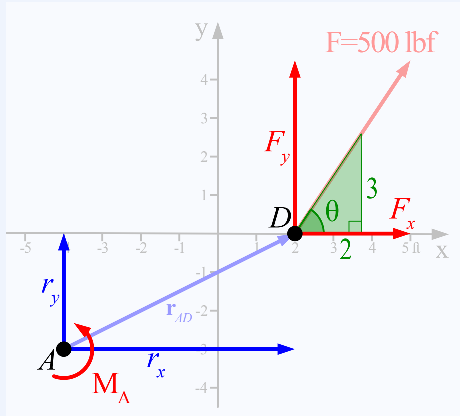
This problem demonstrates four different ways you can solve the problem. The first two methods use vector algebra; the second two take a scalar approach that uses geometry and right-triangle trigonometry. All four methods are mathematically identical.

(a) Find the moment of $\mathbf{F}$ about point A using Varignon's Theorem,

$$\mathbf{M}_A = (\mathbf{r}_x \times \mathbf{F}_y) + (\mathbf{r}_y \times \mathbf{F}_x)\,.$$

Answer. $\mathbf{M_A} = 1664.10\ \text{ft}\cdot\text{lbf}\,(+khat)$

Solution. Varignon's Theorem states that the moment of a force is the sum of the moments of its components. In this example we will determine the vertical and horizontal components of $\mathbf{r}$ and $\mathbf{F}$, then add the cross products of the two perpendicular pairs.



The 3:2 slope of **F** can be expressed as an angle.

$$\theta = \tan^{-1}\frac{3}{2} = 56.3°$$

Find the components of **r** and **F**.

$$\begin{aligned}\mathbf{F} &= 500 \text{ lbf}\langle\cos 56.3°, \sin 56.3°\rangle\\ &= \langle 277.35, 416.025\rangle \text{ lbf}\\ \mathbf{r} &= \langle 6, 3\rangle \text{ ft}\end{aligned}$$

Finally, following Varignon's Theorem, add the cross products of the perpendicular component pairs.

$$\begin{aligned}\mathbf{M}_A &= (\mathbf{r}_x \times \mathbf{F}_y) + (\mathbf{r}_y \times \mathbf{F}_x)\\ &= 6 \text{ ft} \cdot 416.025 \text{ lbf}\,(\mathbf{k}) + 3 \text{ ft} \cdot 277.35 \text{ lbf}\,(-\mathbf{k})\\ &= 1664.1 \text{ ft}\cdot\text{lbf}\,(+\mathbf{k})\end{aligned}$$

Notes:

- When finding the moment of two-dimensional vectors in component form, this is often the preferred method, as it is quick and most find the process intuitive.
- The first cross product, $\mathbf{r}_x \times \mathbf{F}_y$, has a *positive value* because $\mathbf{i}\times\mathbf{j} = +\mathbf{k}$, *not* because you are simply multiplying two postive components.
- The second cross product, $\mathbf{r_y} \times \mathbf{F_x}$, results in a *negative value* because $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$.
- All moments have units of force times distance, in this case [ft · lbf].

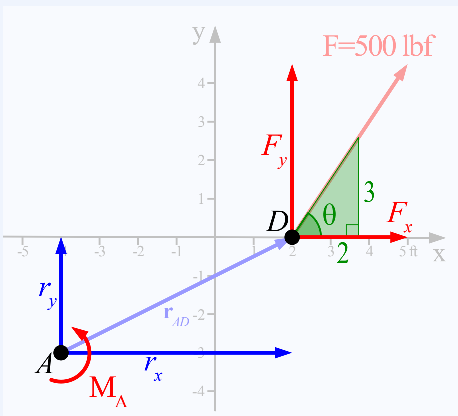
- The overall sign of $\mathbf{M_A}$ determines the final direction. A positive value corresponds to a counterclockwise moment – right thumb out of the page – and a negative value indicates a clockwise moment. See Figure 4.1.2 for the hand diagram.

(b) Find the moment of $\mathbf{F}$ about point A using a vector cross product,

$$\mathbf{M}_A = \mathbf{r} \times \mathbf{F}.$$

Answer. $\mathbf{M_A} = 1664.10\ \text{ft}\cdot\text{lbf}\,(+\mathbf{k})$

Solution.



We can also solve for the moment $\mathbf{M}_A$ using the vector determinant method of Subsection 2.8.3. We can use the values of θ, $\mathbf{r}$, and $\mathbf{F}$ computed in part (a) above. Jumping straight into the vector determinant, we find:

$$\begin{aligned}
\mathbf{M}_A &= \mathbf{r} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & 0 \\ F_x & F_y & 0 \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 3 & 0 \\ 277.35 & 416.025 & 0 \end{vmatrix} \\
&= 6\ \text{ft}\,(416.025\ \text{lbf})\,(\mathbf{k}) + 3\ \text{ft}\,(277.35\ \text{lbf})\,(-\mathbf{k}) \\
&= 1664.1\ \text{ft}\cdot\text{lbf}\,(+\mathbf{k})
\end{aligned}$$

Notes:

- Determinants are a robust way to compute two-dimensional cross products but take a bit more effort than Varignon's Theorem. The math is *exactly the same* either way, which means that Varignon's Theorem is just a two-dimensional shortcut to working through a vector determinant.

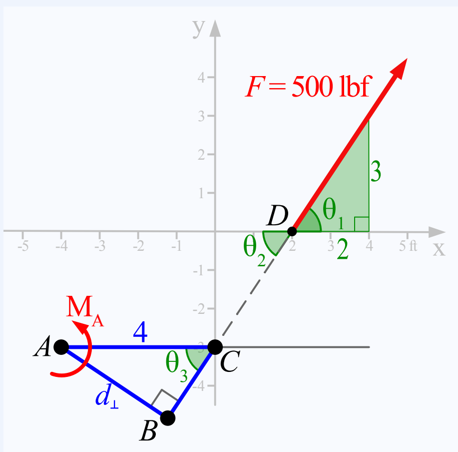
- The signs on the cross-product terms $\mathbf{r}_x \times \mathbf{F}_y$ and $\mathbf{r}_y \times \mathbf{F}_x$ still come from the right-hand rule, and conveniently the process of multiplying diagonals in the determinant takes care of the signs.
- Recognize that the reason we multiply diagonals in a determinant is that we only want to multiply the *perpendicular* components.

(c) Find the moment of $\mathbf{F}$ about point A by finding the perpendicular distance $d_\perp$,

$$M_A = F\, d_\perp.$$

Answer. $\mathbf{M_A} = 1664.10$ ft·lbf $(+\mathbf{k})$

Solution.



This solution requires you to find the perpendicular distance $d_\perp$ between the point A and line-of-action of $\mathbf{F}$. One way to find this distance is shown below.

(a) Draw a moderately large and accurate diagram. Too much confusion has been created by small, inaccurately-drawn diagrams.

(b) Start with the angle θ that you found in Part (a) of this example. The angle opposite θ_1 is θ_2.

(c) Next, using the corresponding angles of parallel lines, transfer θ_2 from the force triangle to triangle ABC as θ_3.

(d) Finally, find $d_\perp$ using the sine function.

$$\begin{aligned} \sin\theta_3 &= \frac{d_\perp}{AC} \\ d_\perp &= AC\,(\sin\theta_3) \\ &= 4\,(\sin 56.3°) \\ &= 3.328 \text{ ft} \end{aligned}$$

(e) Finally, compute the moment about A.

$$
\begin{aligned}
M_A &= F\ d_\perp \\
&= 3.328\ \text{ft}\,(500\ \text{lbf}) \\
&= 1664.10\ \text{ft}\cdot\text{lbf} \\
\mathbf{M}_A &= 1664.10\ \text{ft}\cdot\text{lbf}\,(+\mathbf{k})
\end{aligned}
$$

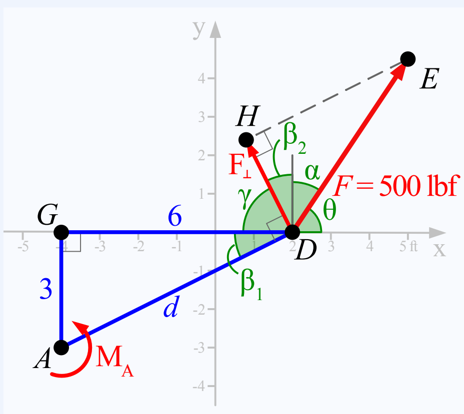
The $(+\mathbf{k})$ direction of $\mathbf{M}_A$ comes from the observation of the right-hand rule, as scalar moment computations are not directional.

(d) Find the moment of $\mathbf{F}$ about point A by finding the perpendicular component of $\mathbf{F}$,

$$M_A = F_\perp\ d.$$

Answer. $\mathbf{M_A} = 1664.10\ \text{ft}\cdot\text{lbf}\,(+\mathbf{k})$

Solution.



This solution requires you to find the portion of force F perpendicular to the moment arm d. One approach to finding $F_\perp$ is shown below.

(a) Draw a large and accurate diagram to assist in finding the distances and angles in this problem.

(b) The next three steps focus on finding the angle $\beta_2 + \alpha$ to help find $\mathbf{F}_\perp$. Using triangle ADG, compute the angle β_1.

$$\beta_1 = \tan^{-1}\left(\frac{3}{6}\right) = 26.565°$$

(c) Next, recognizing that β_1 is measured from horizontal and $F_\perp$ is perpendicular to segment d, then the angle between vertical and $F_\perp$ must also be β, which we'll label β_2. This geometric rule for horizontal: vertical angles of perpendicular lines is also supported by the fact that β_1 and β_2 are both complementary to γ.

(d) The last angle needed is α, which is complimentary to θ.

$$\alpha = 90^\circ - \theta = 90^\circ - 56.31^\circ = 33.69^\circ$$

(e) Find $\mathbf{F}_\perp$ using right triangle DEH.

$$\begin{aligned}\cos(\beta_2 + \alpha) &= \frac{F_\perp}{500 \text{ lbf}} \\ F_\perp &= 500 \text{ lbf}(\cos(26.565^\circ + 33.69^\circ)) \\ F_\perp &= 248.07 \text{ lbf}\end{aligned}$$

(f) Find the length d using the Pythagorean Theorem.

$$d = \sqrt{6^2 + 3^2} = 6.708 \text{ ft}$$

(g) Finally, compute the magnitude M_A and the vector $\mathbf{M_A}$.

$$\begin{aligned}M_A &= F_\perp d \\ &= 248.07 \text{ lbf} \cdot 6.708 \text{ ft} \\ \mathbf{M_A} &= 1664.10 \text{ ft}\cdot\text{lbf}\,(+\mathbf{k})\end{aligned}$$

The counterclockwise direction ($+\mathbf{k}$) comes from the right-hand rule, since scalar moment computations are not directional.

4.5 3D Moments

Key Questions

- Where does the moment arm vector $\mathbf{r}$ start and end?
- Why does Varignon's Theorem give you the same answer as a determinant?
- How can you combine a dot product and a cross product to find the moment about a line?
- Why does a mixed-triple determinant give you a scalar while a cross-product determinant gives you a vector?

The circular arrows we used to represent vectors in two dimensions are unclear in three dimensions, so instead, moments are drawn as arrows and represented by x, y and z components, like force and position vectors. You will sometimes see moments indicated with double arrowheads to differentiate them from force vectors.

In three dimensions, it is usually not convenient to find the moment arm and use equation (4.2.1), so instead we will use the vector cross product, which is easier to apply but less intuitive.

4.5.1 Moment Cross Products

The most robust and general method to find the moment of a force is to use the vector cross product

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}, \tag{4.5.1}$$

where $\mathbf{F}$ is the force creating the moment, and $\mathbf{r}$ is a position vector from the moment center to the line of action of the force. The cross product is a vector multiplication operation and the product is a vector perpendicular to the vectors you multiplied.

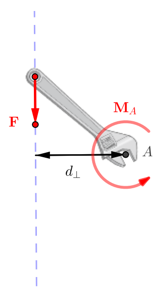


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Figure 4.5.1 Moment cross product. $\mathbf{M} = \mathbf{r} \times \mathbf{F}$

The mathematics of cross products was discussed in Section 2.8, and equation (2.8.1) provides one method to calculate a moment cross products

$$\mathbf{M} = |\mathbf{r}||\mathbf{F}| \sin\theta\, \hat{\mathbf{u}}. \tag{4.5.2}$$

Here, θ is the angle between the two vectors as shown in Figure 4.5.1 above, and $\hat{\mathbf{u}}$ is the unit vector perpendicular to both $\mathbf{r}$ and $\mathbf{F}$ with the direction coming from the right-hand rule. This equation is useful if you know or can find the magnitudes of $\mathbf{r}$ and $\mathbf{F}$ and the angle θ between them. This equation is the vector equivalent of (4.4.2).

Alternately, if you know or can find the components of the position $\mathbf{r}$ and force $\mathbf{F}$ vectors, it's typically easiest to evaluate the moment cross product using the determinant form discussed in Subsection 2.8.3.

$$\begin{aligned}
\mathbf{M} &= \mathbf{r} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \\
&= (r_y F_z - r_z F_y)\,\mathbf{i} - (r_x F_z - r_z F_x)\,\mathbf{j} + (r_x F_y - r_y F_x)\,\mathbf{k}
\end{aligned} \tag{4.5.3}$$

Here, r_x, r_y, and r_z are components of the vector describing the distance from the point of interest to the force. F_x, F_y, and F_z are components of the force. The resulting moment has three components.

$$\begin{aligned}
M_x &= (r_y F_z - r_z F_y) \\
M_y &= (r_x F_z - r_z F_x) \\
M_z &= (r_x F_y - r_y F_x).
\end{aligned}$$

These represent the component moments acting around each of the three coordinate axes. The magnitude of the resultant moment can be calculated using the three-dimensional Pythagorean Theorem.

$$M = |\mathbf{M}| = \sqrt{{M_x}^2 + {M_y}^2 + {M_z}^2} \tag{4.5.4}$$

It is important to avoid three common mistakes when setting up the cross product.

- The order must always be $\mathbf{r} \times \mathbf{F}$, never $\mathbf{F} \times \mathbf{r}$. The moment arm $\mathbf{r}$ appears in the middle line of the determinant and the force $\mathbf{F}$ on the bottom line.
- The moment arm $\mathbf{r}$ must always be measured from the moment center to the line of action of the force. Never from the force to the point.
- The signs of the components of $\mathbf{r}$ and $\mathbf{F}$ must follow those of a right-hand coordinate system.

In two dimensions, r_z and F_z are zero, so (4.5.3) reduces to

$$\begin{aligned}
\mathbf{M} &= \mathbf{r} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & 0 \\ F_x & F_y & 0 \end{vmatrix} \\
&= (r_x F_y - r_y F_x)\, \mathbf{k}.
\end{aligned} \tag{4.5.5}$$

This is just the vector equivalent of Varignon's Theorem in two dimensions, with the correct signs automatically determined from the signs on the scalar components of $\mathbf{F}$ and $\mathbf{r}$.

4.5.2 Moment about a Point

The next two interactives should help you visualize moments in three dimensions.

The first shows the force vector, position vector and the resulting moment all placed at the origin for simplicity. The moment is perpendicular to the plane containing $\mathbf{F}$ and $\mathbf{r}$ and has a magnitude equal to the 'area' of the parallelogram with $\mathbf{F}$ and $\mathbf{r}$ for sides.

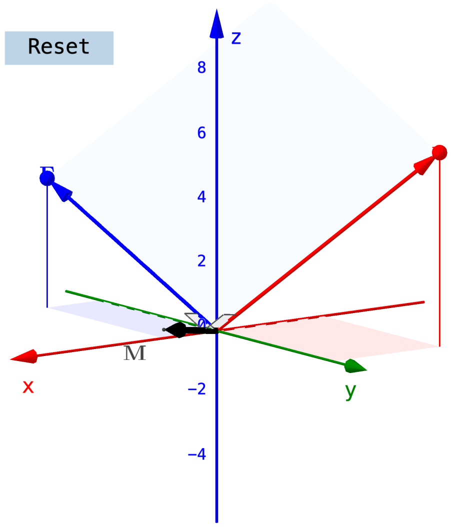

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Figure 4.5.2 Moment about the origin.

The second interactive shows a more realistic situation. The moment center is at arbitrary point A, and the line of action of force $\mathbf{F}$ passes through arbitrary points P_1 and P_2. The position vector $\mathbf{r}$ is the vector from A to a point on the line of action, and the force $\mathbf{F}$ can be slid anywhere along that line.

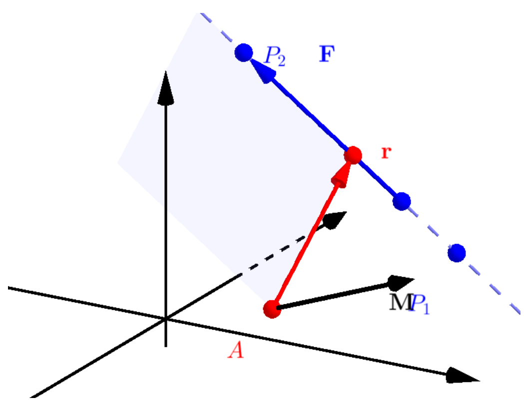

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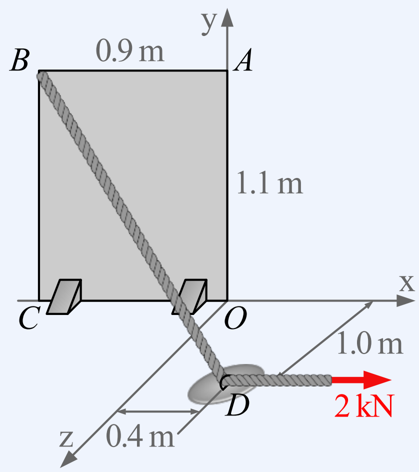
Figure 4.5.3 Moment about an arbitrary point.

Example 4.5.4 3D Moment about a Point.



A thin plate $OABC$ sits in the xy plane. Cable BD pulls with a tension of 2 kN through a frictionless ring at point D. Find the moment caused by the tension force around point O.

Solution. Start the problem by using the position information and tension magnitude to find the force vector $\mathbf{F}_{BD}$. This will be done in three steps:

1. *Find the position vector* **BD**: Find position vectors by either subtracting the start-point coordinates from the end-point coordinates or focusing on the changes in the position components from B to D.

$$\begin{aligned}\mathbf{BD} &= D - B\\ &= (-0.9, 1.1, 0)\text{ m} - (0.4, 0, 1.0)\text{ m}\\ &= \langle 1.3, -1.1, 1\rangle\ \text{ m}\end{aligned}$$

2. *Find the unit vector of* **BD**: Compute a unit vector by dividing **BD** by the total length of BD.

$$\begin{aligned}BD &= |\mathbf{BD}| = \sqrt{1.3^2 + (-1.1)^2 + 1.0^2}\\ &= 1.975\text{ m}\\ \widehat{\mathbf{BD}} &= \frac{\mathbf{BD}}{BD}\\ &= \frac{\langle 1.3, -1.1, 1\rangle\ \text{ m}}{1.975\text{ m}}\\ &= \langle 0.658 - 0.5570.506\rangle\end{aligned}$$

Note that $\widehat{\mathbf{BD}}$ is unitless and is the pure direction of **BD**.

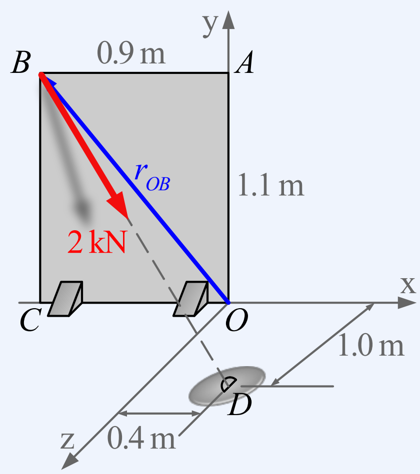
3. *Multiply the unit vector by force magnitude:* Now multiply $\widehat{\mathbf{BD}}$ by the 2 kN force magnitude to find the force components.

$$\mathbf{F}_{BD} = F_{BD}(\widehat{\mathbf{BD}})$$

$$= 2 \text{ kN} \langle 0.658, -0.557, 0.506 \rangle$$
$$= \langle 1.317, -1.114, 1.013 \rangle \text{ kN}$$

Next, find the moment arm from point O to the line of action of the force. There are two obvious options for moment arms, either $\mathbf{r}_{OB}$ or $\mathbf{r}_{OB}$. To demonstrate how both moment arms give the same answer, solutions for both moment arms will be shown.

Option 1: Moment using $\mathbf{r}_{OB}$



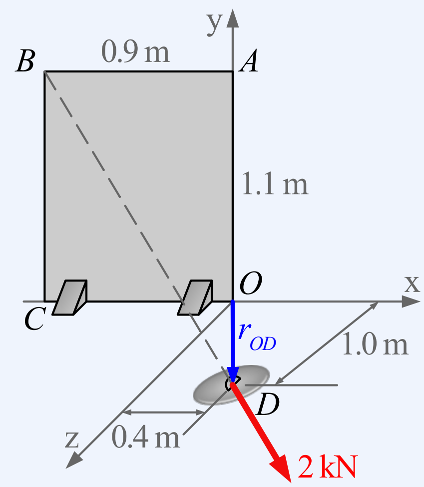
- Moment arm $\mathbf{r}_{OB}$ starts at the point we are taking the moment around, O, and ends at the point B.

$$\begin{aligned}\mathbf{OB} &= B - O \\ &= (-0.9, 1.1, 0) \text{ m} - (0,0,0) \text{ m} \\ &= \langle -0.9, 1.1, 0 \rangle \text{ m}\end{aligned}$$

- Cross $\mathbf{r}_{OB}$ with $\mathbf{F}_{BD}$ to find the moment of $\mathbf{F}_{BD}$ about point O.

$$\begin{aligned}\mathbf{M}_O &= \mathbf{r}_{OB} \times \mathbf{F}_{BD} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.9 & 1.1 & 0 \\ 1.317 & -1.114 & 1.013 \end{vmatrix} \\ &= \langle 1.114, 0.911, -0.446 \rangle \text{ kN}\end{aligned}$$

Option 2: Moment using $\mathbf{r}_{OD}$:



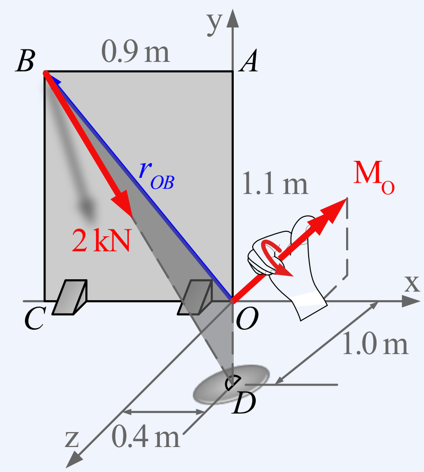
- Moment arm $\mathbf{r}_{OD}$ starts at the point we are taking the moment around, O, and ends at the point D.

$$\begin{aligned}\mathbf{OD} &= D - O \\ &= (0.4, 0, 1.0)\ \text{m} - (0, 0, 0)\ \text{m} \\ &= \langle 0.4, 0, 1.0 \rangle\ \text{m}\end{aligned}$$

- Cross $\mathbf{r}_{OD}$ with $\mathbf{F}_{BD}$ to find the moment of $\mathbf{F}_{BD}$ about point O.

$$\begin{aligned}\mathbf{M_O} &= \mathbf{r}_{OD} \times \mathbf{F}_{BD} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & 0 & 1.0 \\ 1.317 & -1.114 & 1.013 \end{vmatrix} \\ &= \langle 1.114, 0.911, -0.446 \rangle\ \text{kN}\end{aligned}$$

It is worth your effort to compute moments both ways for this example, or another problem, to prove to yourself that the answers work out exactly the same with different moment arms. Technically, you could select a position vector from anywhere on line **BD** and get the correct answer, but $\mathbf{r}_{OB}$ or $\mathbf{r}_{OB}$ are the only two between defined points in this problem.



Drawing $\mathbf{M}_O$, demonstrates that a moment vector direction is both 1) the axis of rotation caused by $\mathbf{T}_{BD}$ around point O, with the moment aligning to your thumb and the moment rotating around your fingers from the right-hand rule and 2) that $\mathbf{M}_O$ is perpendicular to the plane formed by $\mathbf{T}_{BD}$ and $\mathbf{T}_{BD}$. Recall that *all* cross products result in vectors perpendicular to the two crossed vectors.

4.5.3 Moment about a Line

In three dimensions, the moment of a force about a point can be resolved into components about the x, y and z axes. The moment produces a rotational tendency about all three axes simultaneously, but only a portion of the total moment acts about any particular axis.

We are often interested in finding the effect of a moment about a specific line or axis. For example, consider the moment created by a push on a door handle. Unless you push with a force exactly perpendicular to the hinge, only a portion of the total moment you produce will act around the hinge axis and be effective to open the door. The moment we are looking for is the vector projection of the moment onto the axis of interest. Vector projections were first discussed in Subsection 2.7.4.

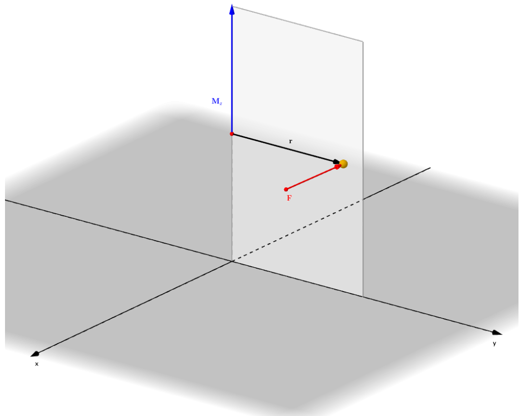


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Figure 4.5.5 Moment on a hinge.

The axis of interest does not need to be a coordinate axis. This interactive shows the projection of moment $\mathbf{M}$ on a line passing through points A and B.

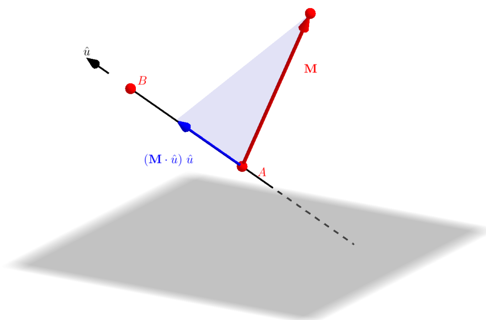

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Figure 4.5.6 Moment of a force about a line

To find the moment of a force about a line, three vectors are required:

- $\hat{\mathbf{u}}$, a unit vector pointing in the direction of the line or axis of interest.
- $\mathbf{r}$, a position vector from any point on the line of interest to any point on the line of action of the force.
- $\mathbf{F}$, the force vector. If you have multiple concurrent forces, you can treat them individually or add them together first and find the moment of the resultant.

With these vectors known, calculating the moment combines skills you already have learned:

- finding the moment of a force about a point using the cross product, (4.5.1).

$$\mathbf{M} = (\mathbf{r} \times \mathbf{F}).$$

- finding the scalar projection of one vector onto another vector using the dot product, (2.7.10)

$$\|\operatorname{proj}_{\mathbf{u}} \mathbf{M}\| = \hat{\mathbf{u}} \cdot \mathbf{M}$$

This combined dot and cross product is a signed scalar value called the **scalar triple product**. A positive sign indicates that the moment vector points in the positive $\hat{\mathbf{u}}$ direction.

- and multiplying a scalar projection by a unit vector to find the vector projection, (2.7.11)

$$\mathbf{M}_{\hat{\mathbf{u}}} = \| \operatorname{proj}_{\mathbf{u}} \mathbf{M} \| \ \hat{\mathbf{u}}. \tag{4.5.6}$$

Carrying these three operations out produces a vector $\mathbf{M}_{\hat{\mathbf{u}}}$ that is the component of moment $\mathbf{M}$ along a line in the $\hat{\mathbf{u}}$ direction.

The scalar triple product can be calculated efficiently in a single step by evaluating a 3×3 determinant consisting of the components of $\hat{\mathbf{u}}$ in the top row, the components of a position vector $\mathbf{r}$ in the middle row, and the components of the $\mathbf{F}$ in the bottom row using the augmented determinant method Figure 2.8.6.

$$\begin{aligned}
\| \operatorname{proj}_u \mathbf{M} \| &= \hat{\mathbf{u}} \cdot (\mathbf{r} \times \mathbf{F}) \\
&= \begin{vmatrix} u_x & u_y & u_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \\
&= (r_y F_z - r_z F_y)\ u_x + (r_z F_x - r_x F_z)\ u_y + (r_x F_y - r_y F_x)\ u_z
\end{aligned}$$

To find the vector projection along the selected axis, multiply this value by the unit vector for the axis, equation (4.5.6).

4.6 Couples

Key Questions

- What makes a couple different than a typical $\mathbf{r} \times \mathbf{F}$ moment?
- Why is a couple considered a pure moment?
- If a couple is applied about the point we are summing moments, does it still need to be included in the sum of moments equation?

The moments we have considered so far were all caused by single forces producing rotation about a moment center. In this section, we will consider another type of moment, called a **couple**.

A couple consists of two parallel forces, equal in magnitude, opposite in direction, and non-coincident. Couples are special because the pair of forces always cancel each other, which means that a couple produces a rotational effect but never translation. For this reason, couples are sometimes referred to as “pure moments.” The strength of the rotational effect is called the **moment of the couple** or the **couple-moment**.

When a single force causes a moment about a point, the magnitude depends on the magnitude of the force and the location of the point. In contrast, the moment of a couple is the same at every point and only depends on the magnitude of the opposite forces and the distance between them.

For example, consider the interactive where two equal and opposite forces with different lines of action form a couple. The moment of this couple is found by summing the moments of the two forces about arbitrary moment center A, applying positive or negative signs for each term according to the right-hand rule. The moment of the couple is always

$$M = Fd_{\perp} \tag{4.6.1}$$

where $d_{\perp}$ is the perpendicular distance between the lines of action of the forces.

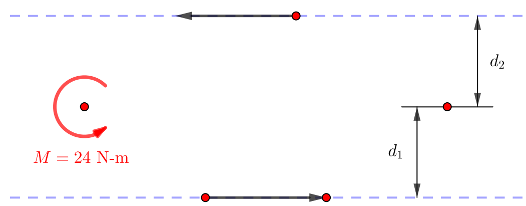

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Figure 4.6.1 Moment of a couple.

In two dimensions, couples are represented by a curved arrow indicating the direction of the rotational effect. Following the right-hand rule, the value will be positive if the moment is counter-clockwise and negative if it is clockwise. In three dimensions, a couple is represented by a normal vector arrow.

When adding moments to find the total or resultant moment, you must include couple-moments as well the $\mathbf{r} \times \mathbf{F}$ moments. In equation form, we could express this as:

$$\Sigma M_P = \Sigma(\mathbf{r} \times \mathbf{F}) + \Sigma(\mathbf{M}_{\text{couple}})$$

Thinking Deeper 4.6.2 Location Independence.

In this section we have shown that couples produce the same moment at every point on the body. This means that the external effect of couples is *location independent*. Because the moment of a couple is location independent, the moment vector is not bound to any particular point and for this reason is a *free vector*.

We will learn in Chapter **??** that moving a couple around on a rigid body does affect the *internal* loads or stresses inside a body, but changing the location of a couple does not change the *external* loading or reactions.

4.7 Equivalent Transformations

Key Questions

- What is an equivalent transformation?
- What are some examples of equivalent transformations?
- What are *external effects*?

An *equivalent transformation* occurs when a loading on an object is replaced with another loading that has the same *external* effect on the object. By external effect, we mean the response of the body that we can see from the outside, with no consideration of what happens to it internally. If the object is a free body, the external effect would be translation and rotation. In statics, since objects are not accelerating, the external effect really means the reactions at the supports required to maintain equilibrium. The external effects will be exactly the same before and after an equivalent transformation.

Equivalent transformation permits us to swap out one set of forces with another one without changing the fundamental physics of the situation. This is usually done to simplify or clarify the situation, or to give you an alternate way to think about, understand, and solve a mechanics problem.

You already know several equivalent transformations although we have not used this terminology before. Here are some transformations you have applied previously.

Vector Addition. When you add forces together using the rules of vector addition, you are performing an equivalent transformation. You can swap out two or more components and replace them with a single equivalent resultant force.

Any number of *concurrent* forces can be added together to produce a single resultant force. By definition, the lines of action of concurrent forces all intersect at a common point. The resultant must be placed at this intersection point in order for this replacement to be equivalent. This is because before and after the replacement, the moment about the intersection point is zero. If the resultant was placed somewhere else, that would not be true.

Replacing a Force with its Component. Resolving forces into components is also an equivalent transformation, as it is the inverse operation of vector addition. The components are usually orthogonal and in the coordinate directions, or in a given plane and perpendicular to it, but any combination of force components that add to the original force is equivalent.

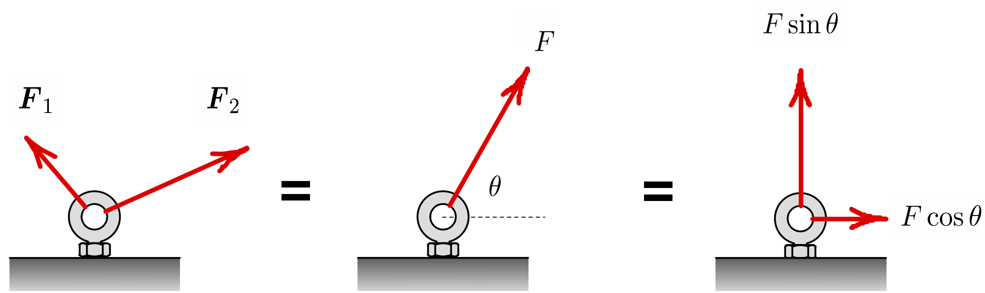


Figure 4.7.1 Equivalent transformations of vectors

In this diagram,

$$\mathbf{F_1} + \mathbf{F}_2 = \mathbf{F} = (F;\theta) = \langle F\cos\theta, F\sin\theta\rangle.$$

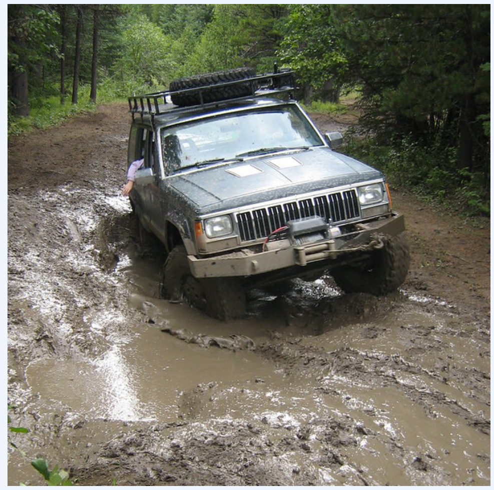
The effects of the force in the x, y and (in three dimensions the z) directions remain the same, and by Varignon's theorem, we know that the moment these forces make about any point will also be the same.

An interesting special case occurs when two forces are equal and opposite and have the same line of action. When these are added together, they cancel out, so replacing these two forces with nothing is an equivalent transformation. The opposite is true as well, so you can make two equal and opposite forces spontaneously appear at a point if you wish.

Thinking Deeper 4.7.2 Internal Effects.

We made a point of saying that equivalent systems of force have the same *external* effect on the body. This implies that there may be some other effects that are not the same. As you will see in Chapter **??**, we sometimes need to consider internal forces and moments. These are the forces inside a body that hold all the parts of the object to each other, otherwise, it would break apart and fail. Although the external effects are the same for all equivalent systems, the internal forces depend on the specifics of how the loads are applied.

Let's imagine that you have gone off-roading and have managed to get your Jeep stuck in the mud. You have two basic options to get it out: you can *pull* it out using the winch on the front bumper, or you can ask your friend to *push* you out with his truck. Both methods (assuming that they apply forces with the same magnitude, direction and line of action) are statically equivalent, and both will equally move your vehicle forward.



The difference is what might happen to your vehicle. With one method there's a danger that you will rip your front bumper off, with the other, you might damage your rear bumper. These are the internal effects and they depend on *where* the equivalent force is applied. These forces are necessary to maintain rigidity and hold the parts of the body together.

Sliding a force along its line of action. Sliding a force along its line of action is an equivalent transformation because sliding a force does not change its magnitude, direction or the perpendicular distance from the line of action to any point, so the moments it creates do not change either. This transformation is called the "Principle of Transmissibility".

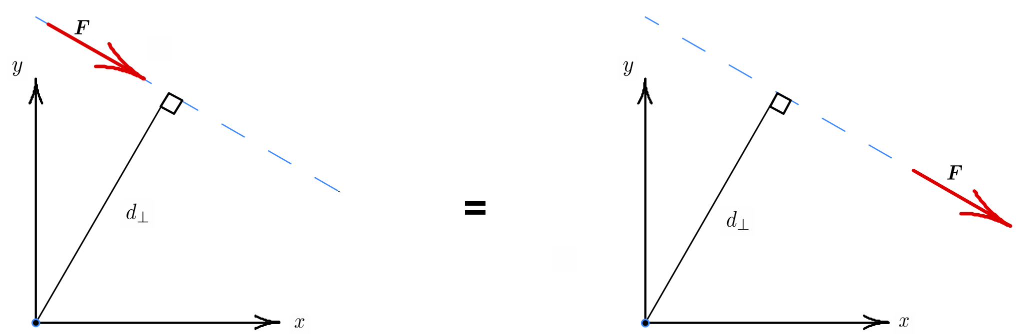


Figure 4.7.3 Sliding a vector along its line of action

Replacing a couple with couple-moment or vice-versa. A *couple*, defined as "two equal and opposite forces with different lines of action," produces a pure turning action that is equivalent to a concentrated moment, called the *couple-moment*. Couples and couple-moments have no translational effect. Couple-moments are free vectors, which means that they are not bound to any point. Their external effect is on the entire body and is the same regardless of where it is applied.

This means that you are free to swap out a couple for its couple-moment, or swap a couple-moment for a couple that has the same moment, and you may put the replacement anywhere on you please and it will still be equivalent.

The diagram shows a series of equivalent transformations of a couple.

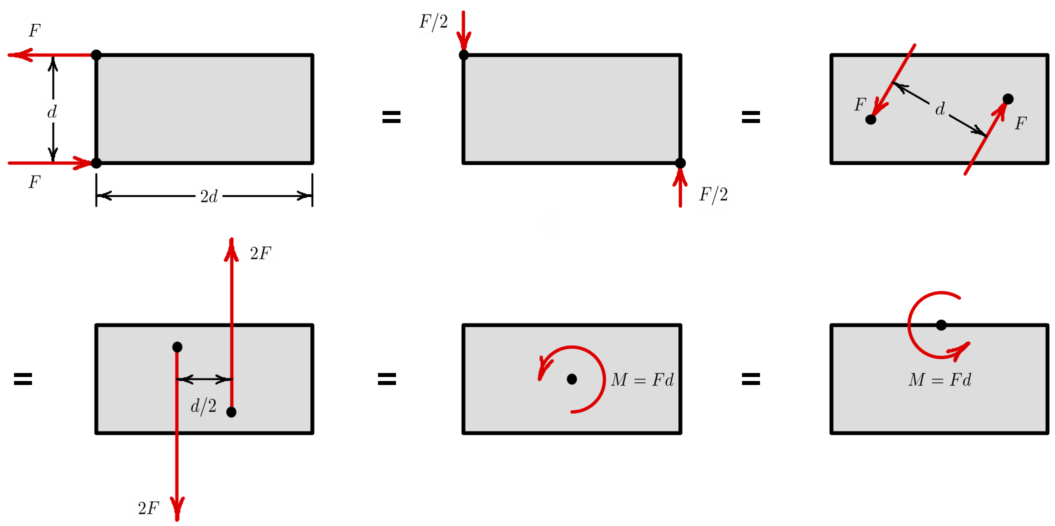


Figure 4.7.4 Equivalent transformations of couples

Concentrated moments are *free vectors*, which you may draw the circular arrow anywhere you like on the body. In other words, moving a concentrated moment from one point to another is an equivalent transformation. Remember though, this equivalence only applies to the external effects. What happens inside the body definitely does depend on the specific point where the moment is applied.

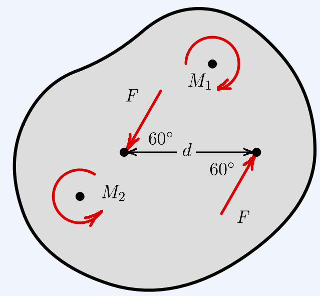
Adding moments to produce a resultant moment. If more than one couple-moment or concentrated moment acts on an object the situation may be simplified by adding them together to produce one *resultant moment*, $\mathbf{M}_R$. The standard rules of vector addition apply.

In two-dimensional problems moments are either clockwise or counter-clockwise, so they may be considered scalar values and added algebraically. Give counter-clockwise moments a positive sign and clockwise moments a negative sign according to the right-hand rule sign convention. If this is done, the sign of the resultant moment will indicate the direction of the net moment. You can use the right-hand rule to establish the direction of the moment vector, which will point into or out of the page.

$$M_R = \Sigma M$$

Example 4.7.5 Equivalent Moment.

Two concentrated moments and a couple are acting on the object shown. Given: $M_1 = 400\ \mathrm{N\cdot m}$, $M_2 = 200\ \mathrm{N\cdot m}$, $F = 40\ \mathrm{N}$ and $d = 2\ \mathrm{m}$.
Replace these with a single, equivalent concentrated moment, and give the magnitude and direction of your result.



Solution. First, replace the couple with an equivalent couple, M_3, the magnitude of which is

$$\begin{aligned} M_3 &= F d_\perp \\ &= F d \sin 60^\circ \\ &= 69.3\ \mathrm{N\cdot m} \end{aligned}$$

By observation, this is a counter-clockwise moment as is M_2. M_1 is clockwise. Summing the scalar magnitudes gives the resultant moment. The signs of the terms are assigned according to the sign convention: positive if counter-clockwise, negative if clockwise.

$$\begin{aligned} M_R &= \Sigma M \\ &= M_1 + M_2 + M_3 \\ &= -400\ \mathrm{N\cdot m} + 200\ \mathrm{N\cdot m} + 69.3\ \mathrm{N\cdot m} \\ &= -130.7\ \mathrm{N\cdot m} \\ \mathbf{M_R} &= 130.7\ \mathrm{N\cdot m\ clockwise} \end{aligned}$$

Resolving a moment into components. For three-dimensional moment vectors, another potential equivalent transformation is to resolve a moment vector into components. These may be orthogonal components in the x, y, and z directions, or components in a plane and perpendicular to it, or components in some other rotated coordinate system.

4.8 Statically Equivalent Systems

Key Questions

- What is an equivalent system?
- What is a resultant force?
- What is a resultant moment?
- Do you have to include both $\mathbf{r} \times \mathbf{F}$ moments and couples to find the resultant moment?
- How can you find the simplest equivalent system?
- When will the simplest equivalent system be a wrench?
- How can you determine if two loading systems are statically equivalent?

A loading system is a combination of load forces and moments that act on an object. It can be as simple as a single force, or as complex as a three-dimensional combination of many force and moment vectors.

You will see that any loading systems may be replaced with a simpler *statically equivalent system* consisting of one *resultant force* at a specific point and one *resultant moment* by performing a series of equivalent transformations. Force system resultants provide a convenient representation for complex force interactions at engineering connections that we will rely on later in a variety of contexts. For now, we will focus on the details of reducing a system to a single force and couple.

Depending on the original loading system, the resultant force, the resultant moment, or both may be zero. If they are both zero, it indicates that the object is in equilibrium under this load condition. If they are non-zero, the supports will need to provide an equal and opposite reaction to put the object into equilibrium.

The resultant force acting on a system, $\mathbf{R}$, can be found from adding the individual forces, $\mathbf{F}_i$, such that

$$\mathbf{R} = \sum \mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + ...$$

The resultant moment, $\mathbf{M}_O$, about a point O, can be found from adding all of the moments $\mathbf{M}$, about that point, including both $\mathbf{r} \times \mathbf{F}$ moments and concentrated moments.

$$\mathbf{M_O} = \sum \mathbf{M}_i = \mathbf{M_1} + \mathbf{M_2} + \mathbf{M_3} + ...$$

It is often more convenient to work with the scalar components of the resultant vectors since they separate the effects in the three coordinate directions.

$$
\begin{aligned}
R_x &= \Sigma F_x & \qquad M_{Ox} &= \Sigma M_x \\
R_y &= \Sigma F_y & \qquad M_{Oy} &= \Sigma M_y \\
R_z &= \Sigma F_z & \qquad M_{Oz} &= \Sigma M_z
\end{aligned}
$$

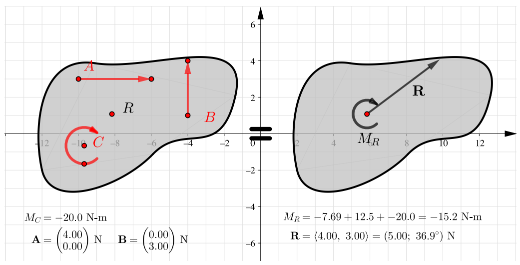


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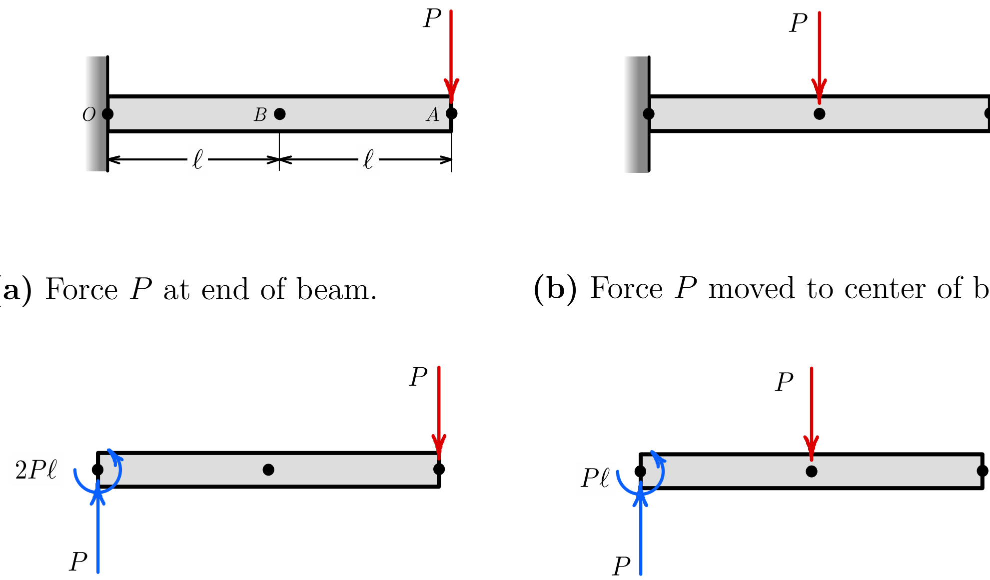
Figure 4.8.1 Statically equivalent systems

Force-Couple Systems. One transformation you might want to make is to move a force to another location. While *sliding* a force along its line of action is fine, *moving* a force to another point changes its line of action and thus its rotational effect on the object, so moving a force to a new line of action is *not* an equivalent transformation.

Consider the cantilever beam below. In diagram (a), the load P is at the end of the beam, and in (b) it has been moved to the center. The external effects are shown in (c) and (d). Although the vertical reaction force is the P in both cases, the reaction moment at point O is $2P\ell$ in the first case and $P\ell$ in the second.



(a) Force P at end of beam.

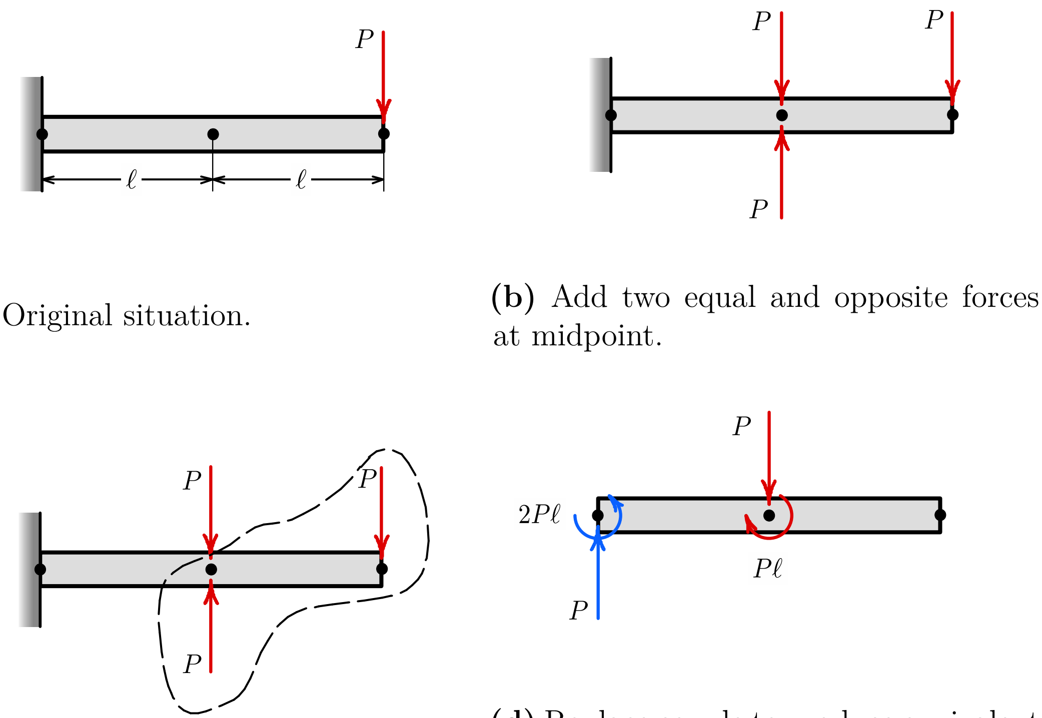
(b) Force P moved to center of beam.

(c) FBD and reactions for (a).

(d) FBD and reactions for (b).

Figure 4.8.2 Moving a force is not an equivalent transformation

You *can* move a force to a new line of action in an equivalent fashion if you add a "compensatory couple" to undo the effect of changing the line of action. This can be accomplished with a series of individual equivalent transformations as shown in the diagram below. To move P to another location, first add two equal and opposite forces where you want the force to be, as in (b). Then recognize the couple you have formed (c), and replace it with an equivalent couple-moment. The result of this process is the *equivalent force-couple system* shown in diagram (d), which is statically equivalent to the original situation in (a).



(a) Original situation.

(b) Add two equal and opposite forces at midpoint.

(c) Recognize couple.

(d) Replace couple to produce equivalent force-couple system, with the same reactions as Figure 4.8.2(c).

Figure 4.8.3 Equivalent Force-couple system

Evaluating the moment at point O was an arbitrary choice. Any other point would give the same result. For example, in the original situation (a) force P makes a clockwise moment $M = P\ell$ about the midpoint. When the force is moved to the center P creates no moment there, so a clockwise compensatory couple with a magnitude of $P\ell$ must be added to maintain equivalence. This is the same result as we found previously (d). The compensatory couple has been drawn centered around the midpoint, but this too is arbitrary because concentrated moments are free vectors and can be placed at any location.

Reduction of a complex system. Any loading system can be reduced to a statically equivalent system consisting of single force and a single moment at a specified point with the following procedure:

1. Determine the resultant moment about the specified point by considering all forces and concentrated moments on the original system.

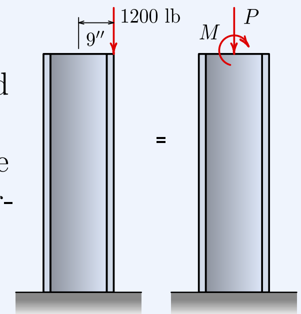
2. Determine the resultant force by adding all forces acting on the original system.

3. Determine the resultant moment about a point in the original system

4. Create the statically equivalent system by replacing all loads with the resultant force and the resultant moment at the selected point.

Example 4.8.4 Eccentric loading.

An vertical column is supporting an eccentric load as shown.
Replace this load with an equivalent force-couple system acting at the center of the beam's top surface.

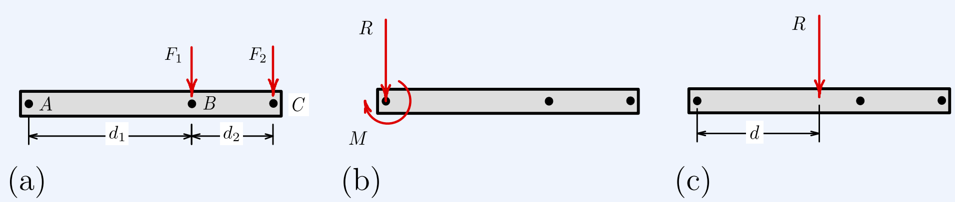


Solution. In order to move the vertical force 9 in to the left, a clockwise couple M must be added to maintain equivalence, where

$$\begin{aligned} M &= Pd \\ &= (1200 \text{ lb})(9 \text{ in}) \\ &= 10,800 \text{ in}\cdot\text{lb} \\ &= 900 \text{ ft}\cdot\text{lb}. \end{aligned}$$

Example 4.8.5 Equivalent Force-couple System.

Replace the system of forces in diagram (a) with an equivalent force-couple system at A.
Replace the force-couple system at A with a single equivalent force and specify its location.



Solution. The original system is shown in (a).
Since the F_1 and F_2 are parallel, the magnitude of the resultant force is just the sum of the two magnitudes and it points down.

$$R = F_1 + F_2$$

The resultant moment about point A is

$$M_A = F_1 d_1 + F_2(d_1 + d_2).$$

To create the equivalent system (b), the resultant force and resultant moment are placed at point A.

The system in (b) can be further simplified to eliminate the moment at M_A, by performing the process in reverse.
In (c) we place the resultant force R a distance d away from point A such that the resultant moment around point A remains the same. This distance can be found using $M = Fd$.

$$d = M_A/R$$

The systems in (a), (b), and (c) are all statically equivalent

In this example, we started with two forces. We have found two different statically equivalent systems; one with a force and a couple, the other with a single force. This latter system is simpler than the original system.

It is important to note that static equivalence applies to external effects only. When determining internal forces, such as the shear and bending moment discussed in Section **??** or when considering non-rigid bodies, the original loading system must be used.

Determining Equivalence. Two complex loading systems are equivalent if they reduce to the same resultant force and the same resultant moment about any arbitrary point.

Two loading systems are statically equivalent if

- The resultant forces are the equal

- The resultant moments about some point are equal

This process is illustrated in the following example.

Example 4.8.6 Finding Statically Equivalent Loads.

Which of the three loading systems shown are statically equivalent?

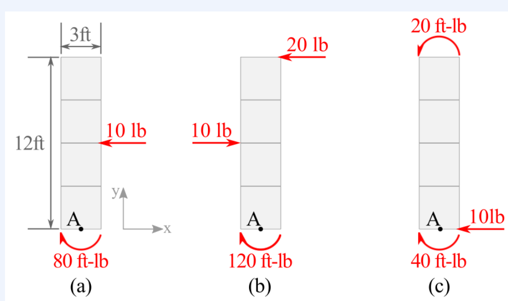


Figure 4.8.7

Solution.

1. *Strategy.*

   Evaluate the resultant force and resultant moment for each case and compare. We choose to evaluate the resultant moment about point A, though any other point would work.

2. *For system (a).*

$$\begin{aligned}\mathbf{R} &= \langle -10, 0\rangle \text{ lb}\\ \mathbf{M_A} &= -80 + 6(10)\\ &= -20 \text{ ft·lb}\end{aligned}$$

3. *For system (b).*

$$\begin{aligned}\mathbf{R} &= \langle -20 + 10, 0\rangle \text{ lb}\\ &= \langle -10, 0\rangle \text{ lb}\\ \mathbf{M}_A &= -120 + 12(20) - 6(10)\\ &= 60 \text{ ft·lb}\end{aligned}$$

4. *For system (c).*

$$\begin{aligned}\mathbf{R} &= \langle -10, 0\rangle \text{ lb}\\ \mathbf{M}_A &= -40 + 20 + 0(10)\\ &= -20 \text{ ft·lb}\end{aligned}$$

Systems (a) and (c) are statically equivalent since **R** and $\mathbf{M}_A$ are the same in both cases. System (b) is not as its resultant moment is different than the other two.

Any load system can be simplified to its resultant force **R**, and resultant couple **M**, acting at any arbitrary point O. There are four common special cases, which are worth highlighting individually.

Concurrent forces. When all forces in a system are concurrent, the resultant moment about their common intersection point will always be zero. We then need only find the resultant force and place it at the point of intersection. The resultant moment about any other point is the moment of the resultant force **R** about that point.

Parallel forces. When all forces in a system are parallel, the resultant force will act in this direction with a magnitude equal to the sum of the individual magnitudes. There will be no moment created about this axis, but we need to find the resultant moment about the other two rectangular axes. That is, if all forces act in the x direction, we need only find the resultant force in the x direction and the resultant moment about the y and z axes.

Coplanar forces. When all forces in a system are coplanar we need only find the resultant force in this plane and the resultant moment about the axis perpendicular to this plane. That is, if all forces exist in the x-y plane, we need only to sum components in the x and y directions to find resultant force **R**, and use these to determine the resultant moment about the z axis. All two-dimensional problems fall into this category.

Wrench resultant. A wrench resultant is a special case where the resultant moment acts around the axis of the resultant force. The directions of the resultant force vector and the resultant moment vector are the same.

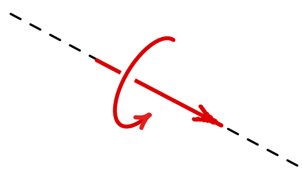


Figure 4.8.8 Wrench Resultant

For example, if the resultant force is only in the x direction and the resultant moment acts only around the x axis, this is an example of a wrench resultant. An everyday example is a screwdriver, where both the resultant force and axis of rotation are in-line with the screwdriver. A wrench resultant is considered positive if the couple vector and force vector point in the same direction, and negative if they point in opposite directions.

Any three-dimensional force-couple system may be reduced to an equivalent wrench resultant even if the resultant force and resultant moment do not initially form a wrench resultant.

To find the equivalent wrench resultant:

1. First, find the resultant force **R** and resultant moment **M** at an arbitrary at arbitrary point, O. These need not act along the same axis.

2. Resolve the resultant moment into scalar components $M_{\|}$ and $M_{\perp}$, parallel and perpendicular to the axis of the resultant force.

3. Eliminate $M_{\perp}$ by moving the resultant force away from point O by distance $d = M_{\perp}/R$

The simplified system consists of moment $\mathbf{M}_{\|}$ and force **R** and acting distance d away from point O. Since **R** and $\mathbf{M}_{\|}$ act along the same axis, the system has been reduced to a wrench resultant. Wrench resultants are the most general way to represent a complex force-couple system, but their utility is limited.

4.9 Exercises (Ch. 4)

Moments	**0/135**
Definition of a moment	0/30 Not attempted
Verignon's Theorem	0/50 Not attempted
Moment of a force about a point	0/20 Not attempted
Semicircle	0/15 Not attempted
Opposing moments using Verignon's theorem	0/20 Not attempted

Couples	**0/40**
Moment of a couple	0/15 Not attempted
Resultant couple	0/15 Not attempted
Couples in equilibrium	0/10 Not attempted

Equivalent Systems	**0/115**
Two parallel forces	0/20 Not attempted
Equivalent force-couple system: Circle	0/30 Not attempted
Equivalent Force-couple system: Bracket	0/35 Not attempted
Resultant of a force-couple system	0/30 Not attempted

Chapter 5

Rigid Body Equilibrium

This chapter will investigate the equilibrium of simple rigid bodies like your book, phone, or pencil. The important difference between rigid bodies and the particles of Chapter 3 is that rigid bodies have the potential to rotate around a point or axis, while particles do not.

For rigid body equilibrium, we need to maintain translational equilibrium with

$$\sum \mathbf{F} = 0 \tag{5.0.1}$$

and also maintain a balance of rotational forces and couple-moments with a new equilibrium equation

$$\sum \mathbf{M} = 0. \tag{5.0.2}$$

5.1 Degree of Freedom

Degrees of freedom refers to the number of independent parameters or values required to specify the *state* of an object.

The *state* of a *particle* is completely specified by its location in space, while the state of a rigid body includes its location in space and also its orientation.

Two-dimensional rigid bodies in the xy plane have three degrees of freedom. Position can be characterized by the x and y coordinates of a point on the object and orientation by angle θ_z about an axis perpendicular to the plane. The complete movement of the body can be defined by two linear displacements Δx and Δy, and one angular displacement $\Delta\theta_z$.

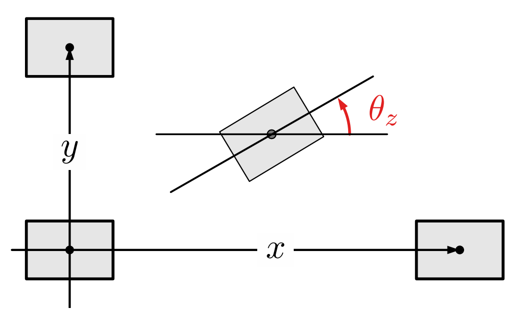


Figure 5.1.1 Two-dimensional rigid bodies have three degrees of freedom.

Three-dimensional rigid bodies have six degrees of freedom, which can be specified with three orthogonal coordinates x, y and z, and three angles of rotation, θ_x, θ_y and θ_z. Movement of the body is defined by three translations Δx, Δy and Δz, and three rotations $\Delta\theta_x$, $\Delta\theta_y$ and θ_z.

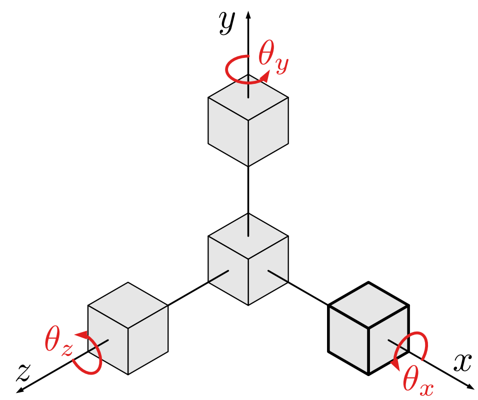


Figure 5.1.2 Three-dimensional rigid body have six degrees of freedom - three translations and three rotations.

For a body to be in static equilibrium, all possible movements must be adequately restrained. If a degree of freedom is not restrained, the body is in an unstable state, free to move in one or more ways. Stability is highly desirable for reasons of human safety, and bodies are often restrained by *redundant* restraints so that if one were to fail, the body would still remain stable. If the restraints correctly interpreted, then equal constraints and degrees of freedom create a stable system, and the values of the reaction forces and moments can be determined using equilibrium equations. If the number of restraints exceeds the number of degrees of freedom, the body is in equilibrium but you will need techniques we won't cover in statics to determine the reactions.

5.2 Free-Body Diagrams

Key Questions

- What are the five steps to create a free-body diagram?
- What are degrees of freedom, and how do they relate to stability?
- Which reaction forces and couple-moments come from each support type?
- What are the typical support force components and couple-moment components that can be modeled from the various types of supports?

Free body diagrams are the tool that engineers use to identify the forces and moments that influence an object. They will be used extensively in statics, and you will use them again in other engineering courses so your effort to master them now is worthwhile. Although the concept is simple, students often need help to draw them correctly.

Drawing a correct free-body diagram is the first and most important step in the process of solving an equilibrium problem. It is the basis for all the equilibrium equations you will write; if your free-body diagram is incorrect, your equations, analysis, and solutions will also be wrong.

A quality free-body diagram is neat, clearly drawn, and contains all the information necessary to solve the equilibrium. You should take your time and think carefully about the free-body diagram before you begin to write and solve equations. A straightedge, protractor and colored pencils all can help. You will inevitably make mistakes that will lead to confusion or incorrect answers; you are encouraged to think about these errors and identify any misunderstandings to avoid them in the future.

Every equilibrium problem begins by drawing and labeling a free-body diagram!

Creating Free Body Diagrams. The basic process for drawing a free-body diagrams is

1. *Select and isolate an object.*

   The "free-body" in free-body diagram means that the body to be analyzed must be free from the supports that are physically holding it in place.

   Simply sketch a quick outline of the object as if it is floating in space disconnected from everything. Do *not* draw free-body diagram forces on top of your problem drawing — the body needs to be drawn free of its supports.

2. *Select a reference frame.*

   Select a right-handed coordinate system to use as a reference for your equilibrium equations. Even if you are using a horizontal x axis and vertical y axis, indicate your coordinate system on your diagram.

   Look ahead and select a coordinate system that minimizes the number of unknown force components in your equations. The choice is technically arbitrary, but a good choice will simplify your calculations and reduce your effort. If you and another student pick different reference systems, you should both get the same answer while expressing your work with different components.

3. *Identify all loads.*

   Add vector arrows representing the applied forces and couple moments of acting on the body. These are often obvious. Include the body's weight if it is non-negligible. If a vector has a known line of action, draw the arrow in that direction; if its sense is unknown, assume one. Every vector should have a descriptive variable name and a clear arrowhead indicating its direction.

4. *Identify all reactions.*

Traverse the perimeter of the object and wherever a support was removed when isolating the body, replace it with the forces and/or couple-moments which it provides. Label each reaction with a descriptive variable name and a clear arrowhead. Again, if a vector's direction is unknown just assume one.

The reaction forces and moments provided by common two-dimensional supports are shown in Figure 5.2.1 and three dimensional support in Figure 5.2.2. Identifying the correct reaction forces and couple-moments coming from supports is perhaps the most challenging step in the entire equilibrium process.

5. *Label the diagram.*

   Verify that every dimension, angle, force, and moment is labeled with either a value or a symbolic name if the value is unknown. Supply the information needed for your calculations, but don't clutter the diagram up with unneeded information. This diagram should be a stand-alone presentation.

Drawing good free-body diagrams is surprisingly tricky and requires practice. Study the examples, think hard about them, do lots of problems, and learn from your errors.

Two-dimensional Reactions. Supports supply reaction forces and moment which prevent bodies from moving when loaded. In the most basic terms, forces prevent translation, and moments prevent rotation.

The reactions supplied by a support depend on the nature of the particular support. For example in a top view, a door hinge allows the door to rotate freely but prevents it from translating. We model this as a frictionless pin that supplies a perpendicular pair of reaction forces, but no reaction moment. We can evaluate all the other physical supports in a similar way to come up with the table below. You will notice that some two-dimensional supports only restrain one degree of freedom and others restrain up to three degrees of freedom. The number of degrees of freedom directly correlates to the number of unknowns created by the support.

The table below shows typical two-dimensional support methods and the corresponding reaction forces and moments supplied each.

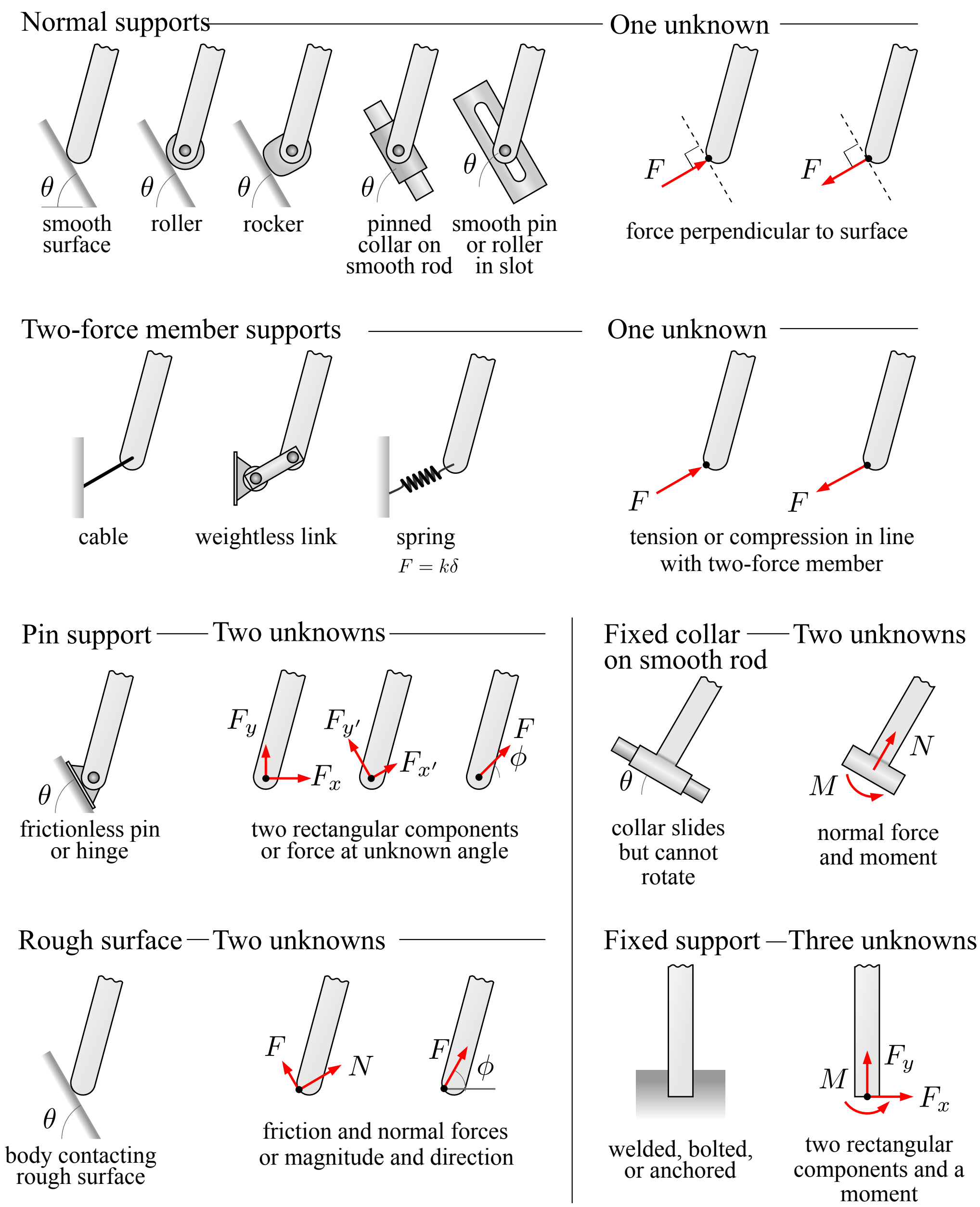


Figure 5.2.1 Table of common two-dimensional supports and their representation on free-body diagrams.

Three-dimensional Reactions. The main added complexity with three-dimensional objects is that there are more possible ways the object can move, and also more possible ways to restrain it. The table below show the types of supports which are available and the corresponding reaction forces and moment. As before, your free-body diagrams should show the reactions supplied by the constraints, not the constraints themselves.

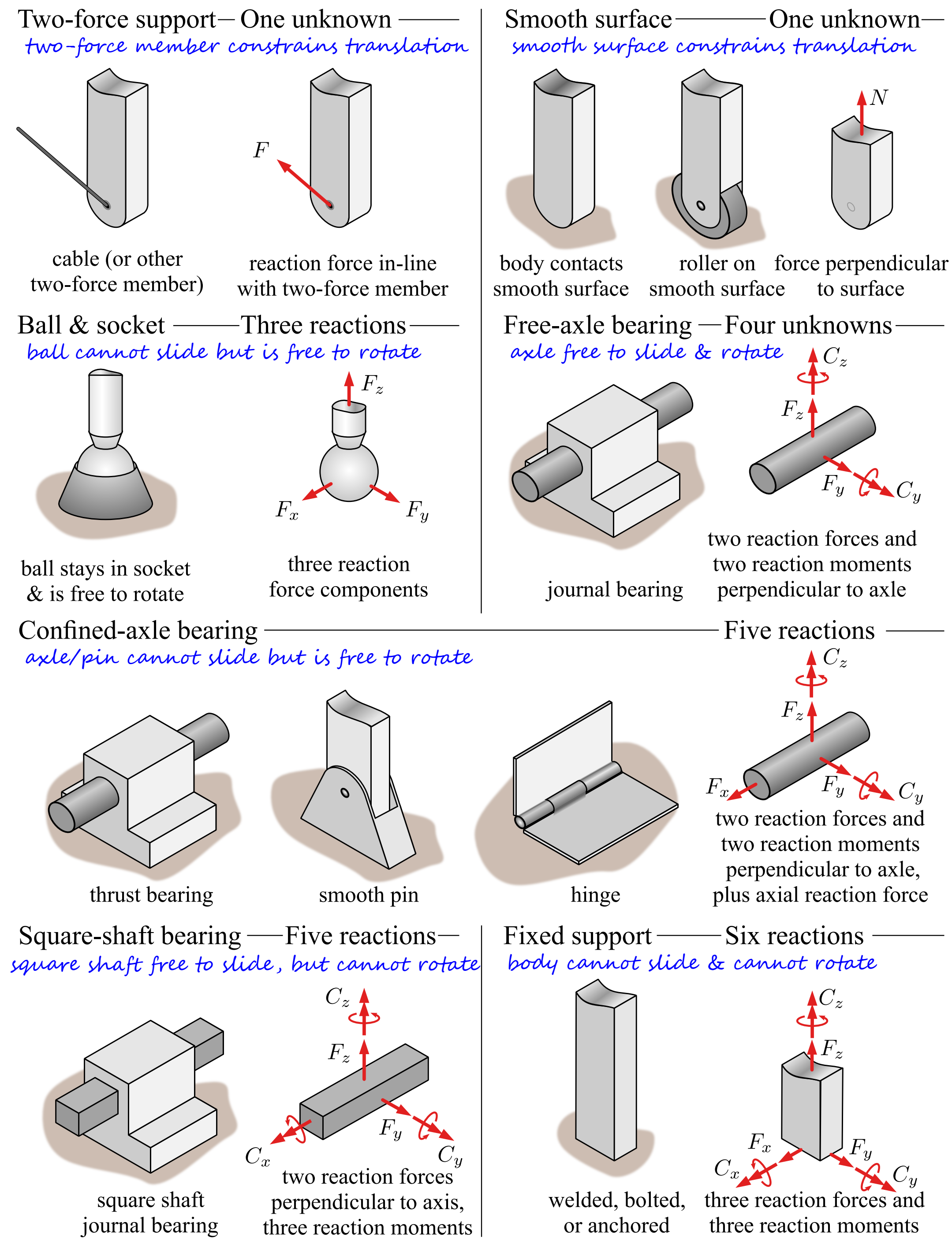


Figure 5.2.2 Table of common three-dimensional supports and their associated reactions.

One new issue we face in three-dimensional problems is that reaction couples may be **available** but not **engaged**.

A support which provides a non-zero reaction is said to be engaged. Picture a crate sitting at rest on a horizontal surface with a cable attached to the top of the crate. If the cable is slack, the reaction of the cable would be available but not *engaged*. Instead, the floor would be supporting the full weight of the crate. If we were to remove the floor, the cable would be engaged and support the weight of the crate.

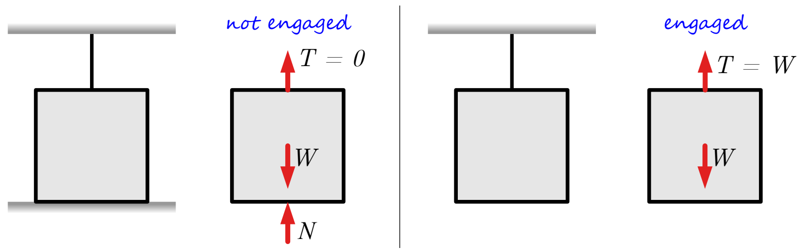


Figure 5.2.3 Available and Engaged reactions.

To get a feel for how reaction couples engage, pick up your laptop or a heavy book and hold it horizontally with your left hand. Can you feel your hand supplying an upward force to support the weight *and* a counter-clockwise reaction couple to keep it horizontal? Now add a similar support by gripping with your right hand. How do the forces and couple-moments change? You should have felt the force of your left hand decrease as your right hand picked up half the weight, and also noticed that the reaction couple from your left hand was no longer needed.

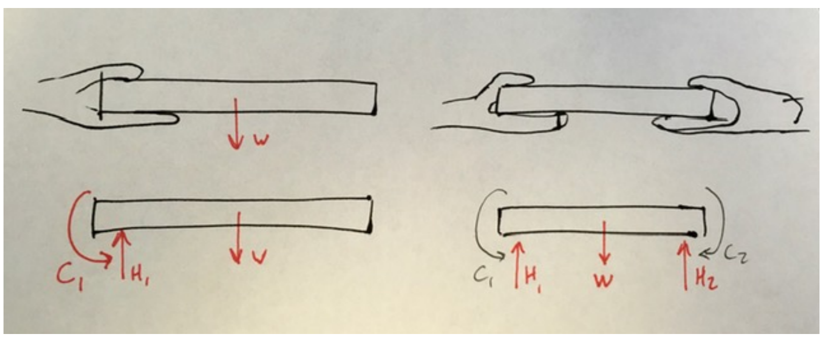


Figure 5.2.4 One hand holding an object versus two hands holding the same object.

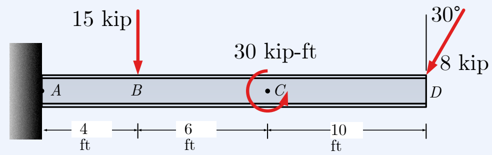
The vertical force in your right hand engaged instead of the couple-moment of your left hand. The reaction couples from both hands are available, but the vertical forces engage first and are sufficient for equilibrium. This phenomena is described by the saying “reaction forces engage before reaction couple-moments”.

Free-Body Diagram Examples. Given that there several options for representing reaction forces and couple-moments from a support, there are different, equally valid options for drawing free-body diagrams. With experience you will learn which representation to choose to simplify the equilibrium calculations.

Possible free-body diagrams for two common situations are shown in the next two examples.

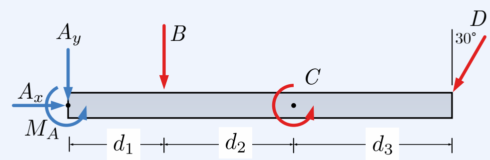
Example 5.2.5 Fixed support.

The cantilevered beam is embedded into a fixed vertical wall at A. Draw a neat, labeled, correct free-body diagram of the beam and identify the knowns and the unknowns.



Solution.

Begin by drawing a neat rectangle to represent the beam disconnected from its supports, then add all the known forces and couple-moments. Label the magnitudes of the loads and the known dimensions symbolically.



Choose the standard xy coordinate system, since it aligns well with the forces.

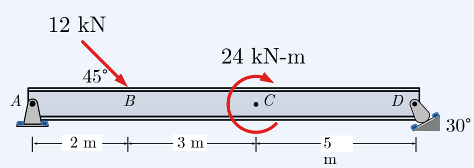
The wall at A is a fixed support which prevents the beam from translating up, down, left or right, or rotating in the plane of the page. These constraints are represented by two perpendicular forces and a concentrated moment, as shown in Figure 5.2.1. Label these unknowns as well.

The knowns in this problem are the magnitudes and directions of moment **C**, forces **B**, and **D** and the dimensions of the beam. The unknowns are the two force components A_x and A_y and the scalar moment M_A caused by the fixed connection. If you prefer, you may represent force **A** as a force of unknown magnitude acting at an unknown direction. Whether you represent it as x and y components or as a magnitude and direction, there are two unknowns associated with force **A**.

The three unknown reactions can be found using the three independent equations of equilibrium we will discuss later in this chapter.

Example 5.2.6 Frictionless pin and roller.

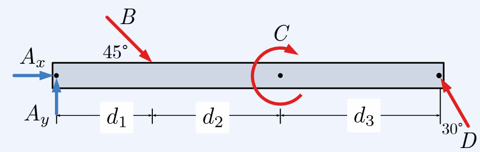
The beam is supported by a frictionless pin at A and a rocker at D. Draw a neat, labeled, correct free-body diagram of the beam and identify the knowns and the unknowns.



Solution. In this problem, the knowns are the magnitude and direction of force **B** and moment **C** and the dimensions of the beam.

The constraints are the frictionless pin at A and the rocker at D. The pin prevents translation but not rotation, which means two it has two unknowns, represented by either magnitude and direction, or by two orthogonal components. The rocker provides a force perpendicular to the surface it rests on, which is 30° from the horizontal. This means that the line of action of force **D** is 30° from the vertical, giving us its direction but not its sense or magnitude

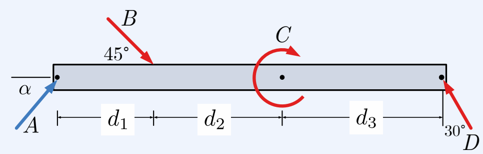
To draw the free-body diagram, start with a neat rectangle to representing the beam disconnected from its supports, then draw and label known force B and moment C and the dimensions.



Add forces A_x and A_y representing vector **A** and force **D** at D, acting 30° from the vertical.

When a force has a known line of action as with force **D**, draw it acting along that line; don't break it into components. When it is not obvious which way a reaction force actually points along its lines of action, just make your best guess and place an arrowhead accordingly. Your calculations will confirm or refute your guess later.

As in the previous example, you could alternately represent force **A** as an unknown magnitude acting in an unknown direction, though there is no particular advantage to doing so in this case.



5.3 Equations of Equilibrium

Key Questions

- What is the definition of static equilibrium?
- How do I choose which are the most efficient equations to solve two-dimensional equilibrium problems?

In statics, our focus is on systems where both linear acceleration **a** and angular acceleration α are zero. These systems are frequently stationary, but could be moving with constant velocity.

Under these conditions Newton's Second Law for translation reduces to

$$\sum \mathbf{F} = 0, \tag{5.3.1}$$

and, Newton's second law for rotation gives the similar equation

$$\sum \mathbf{M} = 0. \tag{5.3.2}$$

The first of these equations requires that all forces acting on an object balance and cancel each other out, and the second requires that all moments balance as well. Together, these two equations are the mathematical basis of this course and are sufficient to evaluate equilibrium for systems with up to six degrees of freedom.

These are vector equations; hidden within each are three independent scalar equations, one for each coordinate direction.

$$\sum \mathbf{F} = 0 \implies \begin{cases} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{cases} \qquad \sum \mathbf{M} = 0 \implies \begin{cases} \Sigma M_x &= 0 \\ \Sigma M_y &= 0 \\ \Sigma M_z &= 0 \end{cases} \tag{5.3.3}$$

Working with these scalar equations is often easier than using their vector equivalents, particularly in two-dimensional problems.

In many cases we do not need all six equations. We saw in Chapter 3 that particle equilibrium problems can be solved using the force equilibrium equation alone, because particles have, at most, three degrees of freedom and are not subject to any rotation.

To analyze rigid bodies, which can rotate as well as translate, the moment equations are needed to address the additional degrees of freedom. Two-dimensional rigid bodies have only one degree of rotational freedom, so they can be solved using just one moment equilibrium equation, but to solve three-dimensional rigid bodies, which have six degrees of freedom, all three moment equations and all three force equations are required.

5.4 2D Rigid Body Equilibrium

Two-dimensional rigid bodies have three degrees of freedom, so they only require three independent equilibrium equations to solve. The six scalar equations of (5.3.3) can easily be reduced to three by eliminating the equations which refer to the unused z dimension. For objects in the xy plane there are no forces acting in the z direction to create moments about the x or y axes, so the reduced set of three equations is

$$\{1\} = \begin{cases} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum M_A &= 0 \end{cases}$$

where the subscript z has been replaced with a letter to indicate an arbitrary moment center in the xy plane instead of a perpendicular z axis.

This is not the only possible set of equilibrium equations. Either force equation can be replaced with a linearly independent moment equation about a point of your choosing [1], so the other possible sets are

$$\{2\} = \begin{cases} \sum F_x &= 0 \\ \sum M_B &= 0 \\ \sum M_A &= 0 \end{cases} \qquad \{3\} = \begin{cases} \sum M_C &= 0 \\ \sum F_y &= 0 \\ \sum M_A &= 0 \end{cases} \qquad \{4\} = \begin{cases} \sum M_C &= 0 \\ \sum M_B &= 0 \\ \sum M_A &= 0 \end{cases}$$

For set four, moment centers A, B, and C must form a triangle to ensure the three equations are linearly independent.

You have a lot of flexibility when solving rigid-body equilibrium problems. In addition to choosing which set of equations to use, you are also free to rotate the coordinate system to any orientation you like, pick different points for moment centers, and solve the equations in any order or simultaneously.

This freedom raises several questions. Which equation set should you choose? Is one choice 'better' than another? Why bother rotating coordinate systems? How do you select moment centers? Students want to know "how to solve the problem," when in reality there are many ways to do it.

The actual task is to choose an efficient approach and carry it out. An efficient solution is one which avoids mathematical complications and makes the problem easy to solve. Complications include unpleasant geometries, unnecessary algebra, and particularly simultaneous equations, which are algebra intensive and error prone.

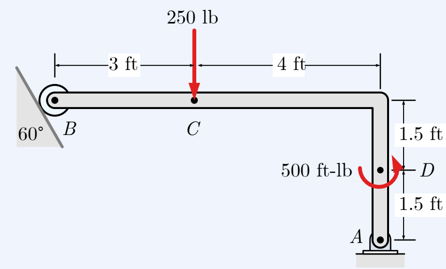
So how do you set up an efficient approach? First, stop, think, and look for opportunities to make the solution more efficient. Here are some recommendations.

1. Equation set one is usually a good choice and should be considered first.
2. Inspect your free-body diagram and identify the unknown values in the problem. These may be magnitudes, directions, angles or dimensions.
3. Align the coordinate system with at least one unknown force.
4. Take moments about the point where the lines of action of two unknown forces intersect, which eliminates them from the equation.
5. Solve equations with one unknown first.

[1] Labels A, B and C in these equations are representative. They don't have to correspond to points A, B and C on your problem.

Example 5.4.1 Pin and Roller.

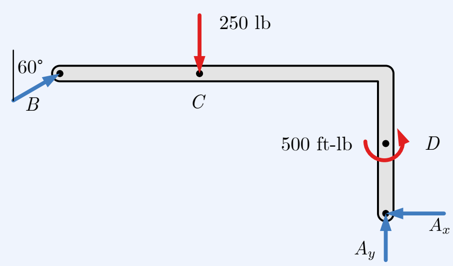
The L-shaped body is supported by a roller at B and a frictionless pin at A. The body supports a 250 lb vertical force at C and a 500 ft·lb couple-moment at D. Determine the reactions at A and B.



This problem will be solved three different ways to demonstrate the advantages and disadvantages of different approaches.

Solution 1.

Solutions always start with a free-body diagram, showing all forces and moments acting on the object. Here, the known loads $C = 250$ lb (down) and $D = 500$ ft·lb (CCW) are red, and the unknown reactions A_x, A_y and B are blue.



The force at B is drawn along its known line-of-action perpendicular to the roller surface, and drawn pointing up and right because that will oppose the rotation of the frame about A caused by load C and moment D. The force at A is represented by unknown components A_x and A_y. The sense of these components is unknown, so we have arbitrarily assigned the arrowheads pointing left and up.

We have chosen the standard coordinate system with positive x to the right and positive y pointing up, and resolved force A into components in the x and y directions.

The magnitude of force B is unknown but its direction is known, so the x and y components of B can be expressed as

$$B_x = B \sin 60^\circ \qquad\qquad B_y = B \cos 60^\circ.$$

We choose to solve equation set $\{A\}$, and choose to take moments about point A, because unknowns A_x and A_y intersect there. Substituting the variables into the equation and solving for the unknowns gives

$$\begin{aligned}\sum F_x &= 0\\ B_x - A_x &= 0\\ A_x &= B\ \sin 60^\circ \qquad (1)\end{aligned}$$

$$\sum F_y = 0$$

$$B_y - C + A_y = 0$$
$$A_y = C - B\ \cos 60° \qquad (2)$$

$$\sum M_A = 0$$
$$-B_x(3) - B_y(7) + C(4) + D = 0$$
$$3B\cos 60° + 7B\sin 60° = 4C + D$$
$$B(3\sin 60° + 7\cos 60°) = 4C + D$$
$$B = \frac{4C + D}{6.098} \qquad (3)$$

Of these three equations only the third can be evaluated immediately, because we know C and D. In equations (1) and (2) unknowns A_x and A_y can't be found until B is known. Inserting the known values into (3) and solving for B gives

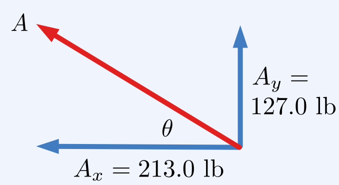
$$\begin{aligned} B &= \frac{4(250) + 500}{6.098} \\ &= \frac{1500\ \text{ft}\cdot\text{lb}}{6.098\ \text{ft}} \\ &= 246.0\ \text{lb} \end{aligned}$$

Now with the magnitude of B known, A_x and A_y can be found with (1) and (2).

$$\begin{aligned} A_x &= B\sin 60° \\ &= 246.0\sin 60° \\ &= 213.0\ \text{lb} \end{aligned}$$

$$\begin{aligned} A_y &= C - B\cos 60° \\ &= 250 - 246.0\cos 60° \\ &= 127.0\ \text{lb} \end{aligned}$$

The positive signs on these values indicate that the directions assumed on the free-body diagram were correct.
The magnitude and direction of force **A** can be found from the scalar components A_x and A_y using a rectangular to polar conversion.



$$A = \sqrt{A_x^2 + A_y^2} = 248.0\ \text{lb}$$

$$\theta = \tan^{-1}\left|\frac{A_y}{A_x}\right| = 30.8°$$

The final values for **A** and **B**, with angles measured counter-clockwise from the positive x axis are

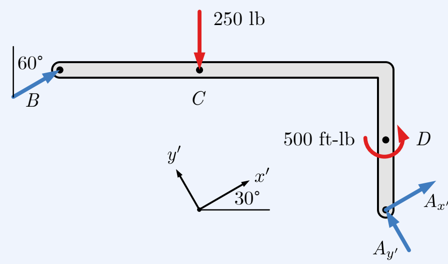
$$\mathbf{A} = 248.0 \text{ lb} \measuredangle 149.2°,$$

$$\mathbf{B} = 246.0 \text{ lb} \measuredangle 30°.$$

This solution demonstrates a fairly standard approach appropriate for many statics problems which should be considered whenever the free-body diagram contains a frictionless pin. Start by taking moments there.

Solution 2.

In this solution, we have rotated the coordinate system 30° to align it with force **B** and also chosen the components of force **A** to align with the new coordinate system.



There is no particular advantage to this approach over the first one, but with two unknown forces aligned with the x' direction, $A_{y'}$ can be found directly after breaking force C into components.

$$\begin{aligned} \sum F_{x'} &= 0 \\ B - C_{x'} + A_{x'} &= 0 \\ A_{x'} &= -B + C \sin 30° \end{aligned} \tag{1}$$

$$\begin{aligned} \sum F_{y'} &= 0 \\ -C_{y'} + A_{y'} &= 0 \\ A_{y'} &= C \cos 30° \end{aligned} \tag{2}$$

$$\begin{aligned} \sum M_A &= 0 \\ -B_x(3) - B_y(7) + C(4) + D &= 0 \\ 3B\cos 60° + 7B \sin 60° &= 4C + D \\ B(3\cos 60° + 7\sin 60°) &= 4C + D \\ B &= \frac{4C + D}{7.56} \end{aligned} \tag{3}$$

Solving equation (2) yields

$$A_{y'} = 216.5 \text{ lb}.$$

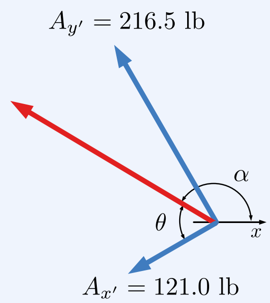
Solving equation (3) yields the same result as previously

$$B = 246.0 \text{ lb.}$$

Substituting B and C into equation (1) yields

$$\begin{aligned} A_{x'} &= -B + C \sin 30^\circ \\ &= -246.0 + 250 \sin 30^\circ \\ &= -121.0 \text{ lb} \end{aligned}$$

The negative sign on this result indicates that our assumed direction for $A_{x'}$ was incorrect, and that force actually points 180° to the assumed direction.
Resolving the $A_{x'}$ and $A_{y'}$ gives the magnitude and direction of force **A**.



$$A = \sqrt{A_{x'}^2 + A_{y'}^2} = 248.0 \text{ lb}$$

$$\theta = \tan^{-1}\left|\frac{A_y}{A_x}\right| = 60.8^\circ$$

$$\alpha = 180^\circ - (\theta - 30^\circ) = 149.2^\circ$$

Again, the final values for **A** and **B**, with angles measured counter-clockwise from the positive x axis are

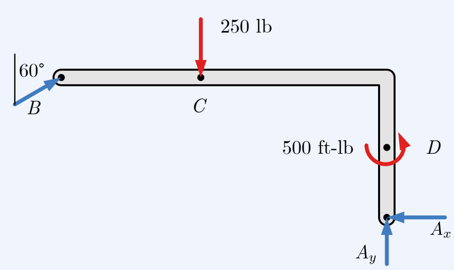
$$\mathbf{A} = 248.0 \text{ lb} \measuredangle 149.2^\circ,$$

$$\mathbf{B} = 246.0 \text{ lb} \measuredangle 30^\circ$$

This approach was slightly more difficult than solution one because of the additional trigonometry involved to find components in the rotated coordinate system.

Solution 3.

For this solution, we will use the same free-body diagram as solution one, but will use three moment equations, about points B, C and D.



$$\sum M_B = 0$$

$$-A_x(3) + A_y(7) - C(3) + D = 0$$

$$-3A_x + 7A_y = 250 \qquad (1)$$

$$\begin{aligned}
\sum M_C &= 0 \\
-A_x(3) + A_y(4) - B_y(3) + D &= 0 \\
-3A_x + 4A_y - 3B\cos 60^\circ &= -D \\
3A_x - 4A_y + 1.5B &= 500 \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\sum M_D &= 0 \\
-A_x(1.5) - B_x(1.5) - B_y(7) + C(4) + D &= 0 \\
1.5A_x + 1.5B\sin 60^\circ + 7B\cos 60^\circ &= 4C + D \\
1.5A_x + 4.799B &= 1500 \qquad (3)
\end{aligned}$$

This set of three equations and three unknowns can be solved with some algebra.
Adding (1) and (2) gives

$$3A_y + 1.5B = 750 \qquad (4)$$

Dividing equation (2) by 2 and subtracting it from (3) gives

$$2A_y + 4.049B = 1250 \qquad (5)$$

Multiplying (4) by 2/3 and subtracting from (5) eliminates A_y and gives

$$3.049B = 750$$

$$B = 246.0 \text{ lb},$$

the same result as before.
Substituting B into (3) gives $A_x = 213.0$ lb, and substituting this into (1) gives $A_y = 127.0$ lb, again the same result as before.
An alternate approach is to set these three equations up for a matrix solution and use technology to do the algebra, as done here with Sage.

$$\begin{bmatrix} -3 & 7 & 0 \\ 3 & -4 & 1.5 \\ 1.5 & 0 & 4.799 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ B \end{bmatrix} = \begin{bmatrix} 250 \\ 500 \\ 1500 \end{bmatrix}$$

```
A = Matrix([[-3,7,0],[3,-4, 1.5],[1.5,0,4.799]])
B = vector([250, 500, 1500])
x = A.solve_right(B)
x
```

```
(213.020662512299, 127.008855362414, 245.982289275172)
```

This is a good example of an inefficient solution because of all the algebra involved. The issue here was the poor choice of B, C and D as moment centers. Whenever possible you should take moments about a point where the line of action of two unknowns intersect as was done in solution one. This gives a moment equation which can be solved immediately for the third unknown.

5.5 3D Rigid Body Equilibrium

Key Questions

- What are the similarities and differences between solving two-dimensional and three-dimensional equilibrium problems?
- Why are some three-dimensional reaction couple-moments "available but not engaged"?
- What kinds of problems are solvable using linear algebra?

Three-dimensional systems are closer to reality than two-dimensional systems and the basic principles to solving both are the same, however they are generally harder solve because of the additional degrees of freedom involved and the difficulty visualizing and determining distances, forces and moments in three dimensions.

Three-dimensional problems are usually solved using vector algebra rather than the scalar approach used in the last section. The main differences are that directions are described with unit vectors rather than with angles, and moments are determined using the vector cross product rather scalar methods. Because they have more possible unknowns it is harder to find efficient equations to solve by hand. A problem might involve solving a system of up to six equations and six unknowns, in which case it is best solved using linear algebra and technology.

Resolving Forces and Moments into Components. To break two-dimensional forces into components, you likely used right-triangle trigonometry, sine and cosine. However, three-dimensional forces will likely need to be broken into components using Section 2.5.

When summing moments, make sure to consider both the $\mathbf{r} \times \mathbf{F}$ moments and also the couple-moments with the following guidance:

1. First, choose any point in the system to sum moments around.
2. There are two general methods for summing the $\mathbf{r} \times \mathbf{F}$ moments. Both techniques will give you the same set of equations.

(a) *Sum moments around each axis.*

For relatively simple systems with few position and force vector components, you can find the cross product for each non-parallel position and force pair. Using this method requires you to resolve the direction of each cross product pair using the right-hand rule as covered in Chapter 4. Recall that there are up to six pairs of non-parallel components that you need to consider.

(b) *Sum all moments around a point using vector determinants.*

Choose a point in the system which is on the line of action of as many forces as possible, then set up each cross product as a determinant. After computing the components coming from each determinant, combine the x, y, and z terms into each of the $\Sigma\mathbf{M}_x = 0$, $\Sigma\mathbf{M}_y = 0$, and $\Sigma\mathbf{M}_z = 0$ equations.

3. Finally, add the components of any couple-moments into the corresponding $\Sigma\mathbf{M}_x = 0$, $\Sigma\mathbf{M}_y = 0$, and $\Sigma\mathbf{M}_z = 0$ equations.

Solving for unknown values in equilibrium equations. Once you have formulated $\Sigma\mathbf{F} = 0$ and $\Sigma\mathbf{M} = 0$ equations in each of the x, y and z directions, you could be facing up to six equations and six unknown values.

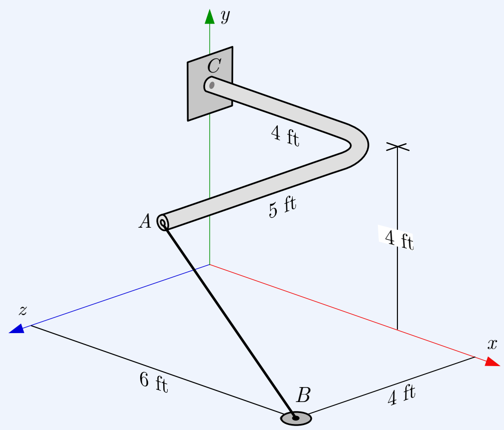
Frequently these simultaneous equation sets can be solved with substitution, but it is often be easier to solve large equation sets with linear algebra. Note that the adjective “linear” specifies that the unknown values must be linear terms, which means that each unknown variable cannot be raised to a exponent, be an exponent, or buried inside of a sin or cos function. Luckily, most unknowns in equilibrium are linear terms, except for unknown angles. If you are not familiar with the use of linear algebra matrices to solve simultaneously equations, search the internet for *Solving Systems of Equations Using Linear Algebra* and you will find plenty of resources.

No matter how you choose to solve for the unknown values, any numeric values which come out to be negative indicate that your initial hypothesis of that vector’s sense was incorrect.

Three-dimensional Equilibrium Examples.

Example 5.5.1 3D Bent Bar.

The bent bar shown is held in a horizontal plane by a fixed connection at C while cable AB exerts a 500 lb force on point A.
Given $A = (4, 4, 5)$ $B = (6, 0, 4)$ and $C = (0, 4, 0)$.

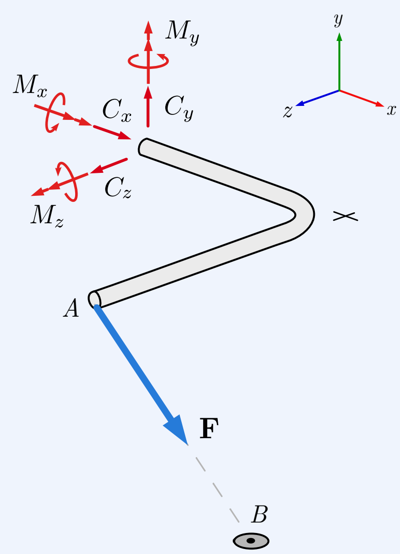


Find the reaction force **C** and concentrated moment **M** with components M_x, M_y and M_z required to hold the bar in this position under this condition,

Solution.

1. *Draw a free-body diagram.*

   As always, begin by drawing a free-body diagram.



2. *Determine the force acting at point A in Cartesian form.*

   The force of the cable acts from A to B. This direction is described by the displacement vector from A to B

$$\mathbf{r}_{AB} = (2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}) \text{ ft}$$

   or the corresponding unit vector

$$\lambda_{AB} = \frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|}$$

$$= \frac{2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}}{\sqrt{(2)^2 + (-4)^2 + (-1)^2}}$$
$$= \frac{2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}}{\sqrt{21}}.$$

Multiplying the unit vector by the cable tension gives the force acting on A as a three-dimensional Cartesian force vector

$$\begin{aligned}\mathbf{F} &= \lambda_{AB} T \\ &= \left(\frac{2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}}{\sqrt{21}}\right) 500 \text{ lb} \\ &= (2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}) \left(\frac{500}{\sqrt{21}}\right) \text{ lb}\end{aligned}$$

$$\mathbf{F} = (218\mathbf{i} - 436\mathbf{j} - 109\mathbf{k}) \text{ lb}.$$

3. *Determine the moment about C.*

   The moment about point C is found with the cross product (4.5.1) where the moment arm is the displacement vector from C to A.

$$\mathbf{r}_{CA} = (4\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}) \text{ ft}$$

$$\begin{aligned}\mathbf{M}_C &= \mathbf{r}_{CA} \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 5 \\ 2 & -4 & -1 \end{vmatrix} \left(\frac{500}{\sqrt{21}}\right)\end{aligned}$$

$$\mathbf{M}_C = (2182\mathbf{i} + 1528\ \mathbf{j} - 1746\mathbf{k}) \text{ ft}\cdot\text{lb}$$

4. *Apply the equations of equilibrium.*

$$\Sigma \mathbf{F} = 0 \qquad \begin{cases} \Sigma F_x = 0 : & C_x + F_x = 0 \\ & C_x = -218 \text{ lb} \\ \Sigma T_y = 0 : & C_y - F_y = 0 \\ & C_y = +436 \text{ lb} \\ \Sigma T_z = 0 : & C_z - F_z = 0 \\ & C_z = +109 \text{ lb} \end{cases}$$

$$\Sigma \mathbf{M} = 0 \qquad \begin{cases} \Sigma M_x = 0: & M_x + M_{Cx} = 0 \\ & M_x = -2180 \text{ ft}\cdot\text{lb} \\ \Sigma M_y = 0: & M_y + M_{Cy} = 0 \\ & M_y = -1530 \text{ ft}\cdot\text{lb} \\ \Sigma M_z = 0: & M_z + M_{Cz} = 0 \\ & M_z = +1750 \text{ ft}\cdot\text{lb} \end{cases}$$

The resulting vector equations for the reaction force **C** and reaction moment **M** are

$$\mathbf{C} = (-218\mathbf{i} + 436\mathbf{j} + 109\mathbf{k}) \text{ lb}$$
$$\mathbf{M} = (-2180\mathbf{i} - 1530\mathbf{j} + 1750\mathbf{k}) \text{ ft}\cdot\text{lb}.$$

5.6 Stability and Determinacy

Key Questions

- What does *stable* mean for a rigid body?
- What does *determinate* mean for a rigid body?
- Does stability depend on the external loads or only on the reactions?
- How can I tell if a system is determinate?
- How can I decide if a problem is both stable and determinate, which makes it solvable statics?

Determinate vs. Indeterminate. A static system is **determinate** if it is possible to find the unknown reactions using the methods of statics, that is, by using equilibrium equations, otherwise it is **indeterminate**.

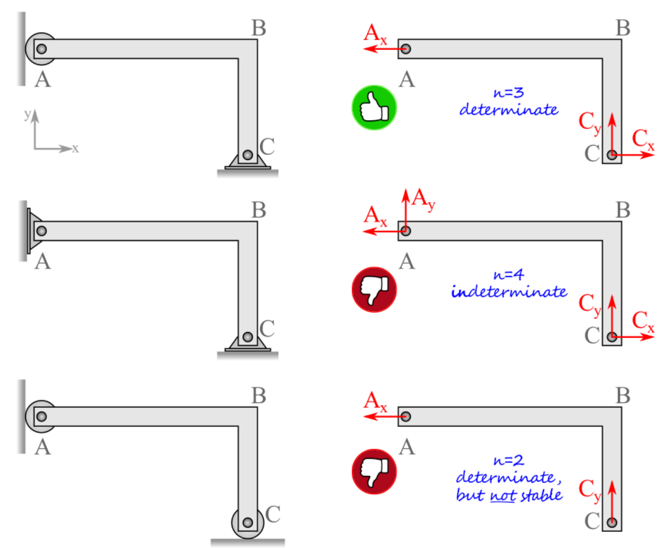
In order for a system to be determinate the number of unknown force and moment reaction components must be less than or equal to the number of independent equations of equilibrium available. Each equilibrium equation derives from a degree of freedom of the system, so there may be no more unknowns than degrees of freedom. This means that we can determine no more than three unknown reaction components in two-dimensional systems and no more than six in three-dimensional systems.

An indeterminate system with fewer reaction components than degrees of freedom is **under-constrained** and therefore unstable. On the other hand, if there are more reaction components than degrees of freedom, the system is both **over-constrained** and **indeterminate**. In terms of force and moment equations, there are more unknowns than equilibrium equations so they can't all be determined. This is not to say that it is impossible to find all reaction force on an over-constrained system, just that you will not learn how to find them in this course.

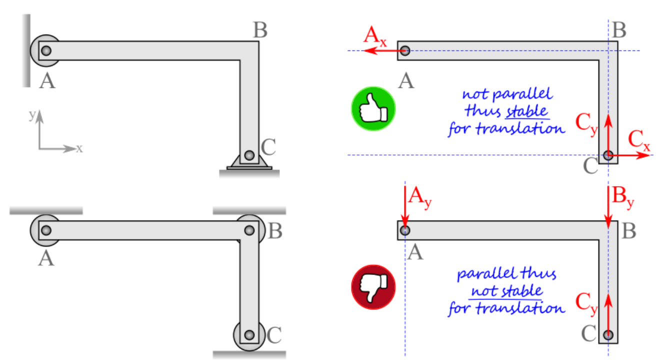
Stable vs. Unstable. A body in equilibrium is held in position by its supports, which restrict the body's motion and counteract the applied loads. When there are sufficient supports to restrain a body from moving, we say that the body is **stable**. A stable body is prevented from translating and rotating in all directions. A body which *can* move is **unstable** even if it is not currently moving, because the slightest change in load may take it out of equilibrium and initiate motion. Stability is loading independent i.e. a stable body is stable for *any* loading condition.

Rules to Validate a Stable and Determinate System. There are three rules to determine if a system is both stable and determinate. While, the rules below can technically be checked in any order, they have been sorted from the quickest to the most time consuming to speed up your analysis.

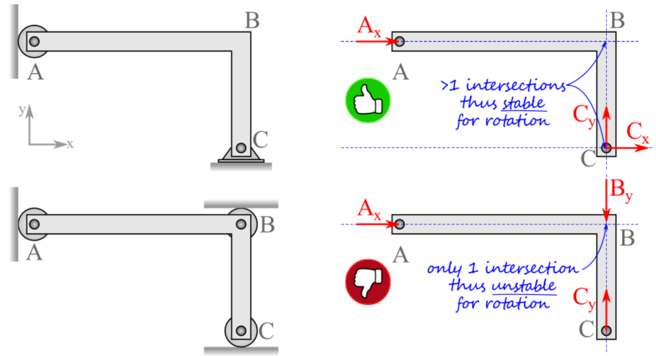
Rule 1: Are there exactly three reaction components on a two-dimensional body? If YES, the system is determinate.
If NO, the system is indeterminate or not stable.



Rule 2: Are all the reaction force components parallel to one another?
If YES, the system is unstable for translation.
If NO, the system is stable for translation.



Rule 3: Do the lines of action of the reaction forces intersect at a single point?
If YES, the system is unstable for rotation about the single intersection point.
If NO, the system is stable for rotation.



5.7 Equilibrium Examples

You can use these interactives to explore how the reactions supporting rigid bodies are affected by the loads applied. You can use the equations of equilibrium to solve for the unknown reactions, and check your work.

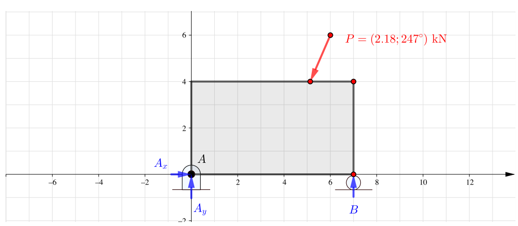


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Figure 5.7.1 Rigid body Equilibrium

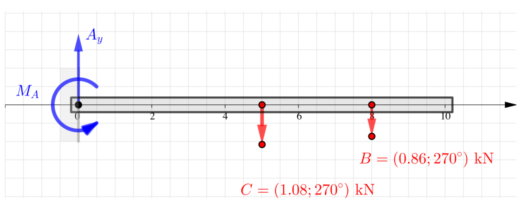

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Figure 5.7.2 Cantilever beam

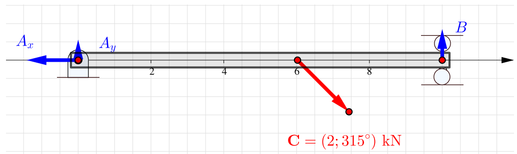

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Figure 5.7.3 Beam with concentrated load

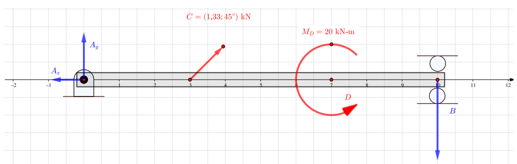

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Figure 5.7.4 Beam with concentrated force and couple moment

5.8 Exercises (Ch. 5)

Tension in a cable	0/4 Not attempted
Raise pole	0/30 Not attempted
Cantilever beam	0/35 Not attempted
EquBeam and pulley	0/40 Not attempted
Bell crank	0/30 Not attempted
Car on a hill	0/20 Not attempted
Hand truck	0/20 Not attempted
Triangle	0/50 Not attempted
Truss	0/60 Not attempted
Triangle	0/35 Not attempted
Beam with angled load	0/20 Not attempted

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Chapter 6

Equilibrium of Structures

In this chapter you will conduct static analysis of multi-body structures. Broadly defined, a structure is any set of interconnected rigid bodies designed to serve a purpose. The parts of the structure may move relative to one another, like the blades of scissors, or they may be fixed relative to one another, like the structural members of bridge.

Analysis of structures involves determining all forces acting on and between individual members of the structure. Fundamentally there is nothing new here; the techniques you have already learned apply, however structures tend to have more unknown forces, and so are more involved and provide more opportunities for error than the problems you have previously encountered. Correct free-body diagrams and careful work are required, as always.

6.1 Structures

Structures fall into three broad categories: trusses, frames, and machines, and you should be able to identify which is which.

A **truss** is a multi-body structure made up of long slender members connected at their ends in triangular subunits. Truss members carry axial forces only. Trusses are commonly used for spanning large distances without interruption: bridges, roof systems, stadiums, aircraft hangers, auditoriums for example. They are also used for crane booms, radio towers and the like. Trusses are lightweight and relatively strong. Over the years many unique truss designs have been developed and are often named after the original designer.

A **frame** is a multi-part, rigid, stationary structure primarily designed to support some type of load. A frame contains at least one multi-force member, which a truss never has. This means that, unlike trusses, frame members must support bending moments as well shear and normal forces. Many common items can be considered frames. Some examples: building structure, bike frames, ladders, scaffolding, and more.

A **machine** is very similar to a frame, except that it includes some moving parts. The purpose of a machine is usually to provide a mechanical advantage and multiply forces. Pliers, scissors jacks, automobile suspensions, construction equipment are all examples of machines.

Figure 6.1.1 Scissors and bridges are examples of engineering structures. Scissors are a machine with three interconnected parts. The bridge is a truss.

Solving a structure means determining all forces acting on all of its parts. The solution typically begins by determining the global equilibrium of the entire structure, then breaking it into parts and analyzing each separate part. The specific process will depend on the type of structure, but will always follow the principles covered in the previous chapters.

Two-force Members. Many structures contain at least one two-force member, and trusses consist of two-force members exclusively. Recall from Subsection 3.3.3 that a two-force body is an object subjected to exactly two forces. Two-force members are not required to be slender or straight, but can be recognized because they connect to other bodies or supports at exactly two points, and have no other loading unless it is also applied at those points.

Identifying two-force members is helpful when solving structures because they automatically establish the line of action of the two forces. In order for a two-force body to be in equilibrium, the forces acting on it must be equal in magnitude, opposite in direction, and have a line-of-action passing through the point where the two forces are applied. Since these points are known, the direction of the line-of-action is readily found.

The common way to express the force of a two-force member is with a magnitude and a sense, where the sense is either tension or compression. If the two forces tend to stretch the object we say it is in *tension*; if they act the other way and squash the object, it is in *compression*. The usual approach is to assume that a two-force member is in tension, then draw the free-body diagram and write the equilibrium equations accordingly. If the analysis shows that the forces are negative then they actually act with the opposite sense, i.e. compression.

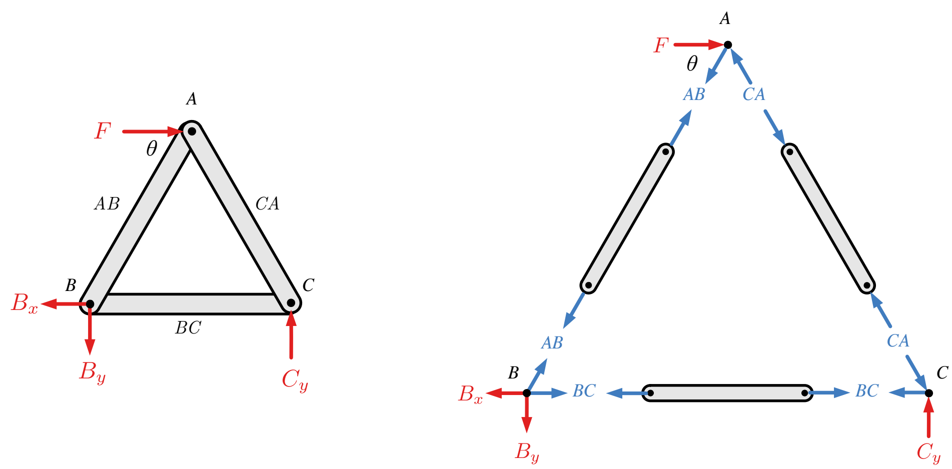
Figure 6.1.2 Two-force members in tension and compression.

6.2 Interactions between members

When analyzing structures we are dealing with multi-body systems, and need to recall Newton's 3rd Law, "For every action, there is an equal and opposite reaction."

This law applies to multi-body systems wherever one body connects to another. At any interaction point, forces are transferred from one body to the interacting body as equal and opposite action-reaction pairs. These forces cancel out and are invisible when the structure is intact. Only when we cut through a member or joint in the isolation step of creating a free-body diagram, do we expose the interaction forces. When drawing free-body diagrams of the components of structures, it is critically important to represent these action-reaction pairs consistently. You may assume either direction for one, but the other *must* be equal and opposite.

For example, look at the members and joints in the truss below. Diagram (a) shows the truss members held together by pins at A, B, and C. The forces holding the parts together cancel and are not shown. In the 'exploded' view (b), the parts have been separated and the action-reaction pairs are exposed. Member AB is in tension, and the forces acting on it, also called AB, oppose each other and tend to stretch the member. These stretching forces are accompanied by equal and opposite forces, also called AB acting on pins A and B. Tensile forces BC and compressive forces CA behave similarly.

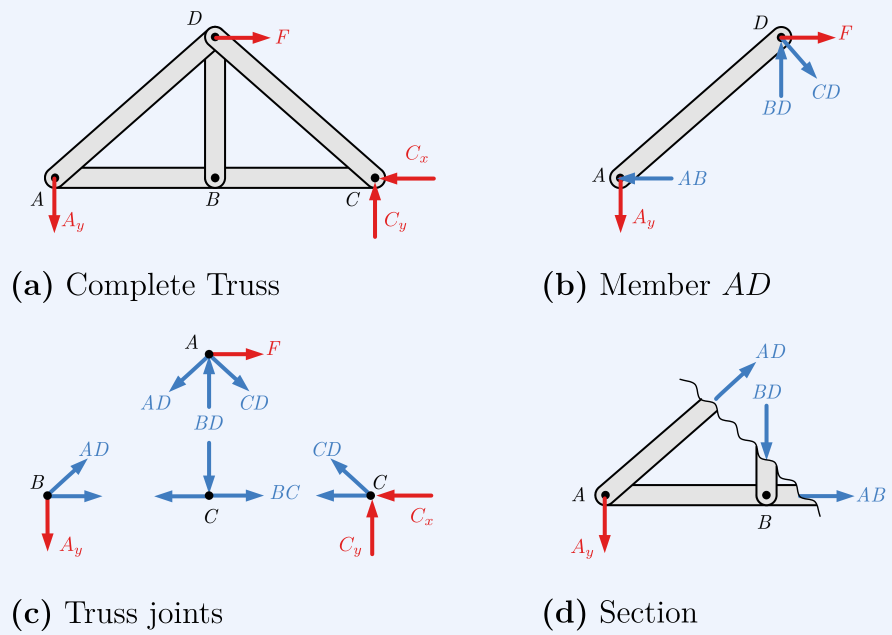


(a) Whole Truss **(b)** Exploded

Figure 6.2.1 External load and global reactions in red. Internal action-reaction pairs in blue.

Thinking Deeper 6.2.2 Multi-body systems.

When a multipart structure is in equilibrium, each part of the structure is also in equilibrium. For example in the truss below, each member of the truss, each joint, and each portion of the truss is also in equilibrium. This continues all the way down to the atoms of the structure. This universal equilibrium across spatial scales is one of the governing principles which allows us to break multi-body systems into smaller solvable parts.



(a) Complete Truss **(b)** Member AD

(c) Truss joints **(d)** Section

Figure 6.2.3 Possible free-body diagrams

You will see in this chapter that we have the freedom to isolate free-body diagrams at any scale to expose our target unknowns.

6.2.1 Load Paths

Load paths can help you think about structural systems. Load paths show how applied forces like the floor load in the image below pass through the interconnected members of the structure until they end up at the fixed support reactions. All structural systems, whether non-moving frames or moving machines have some sort of load path. When analyzing all structures, you computationally move from known values through the interconnected bodies of the system, following the load path, solving for unknowns as you go.

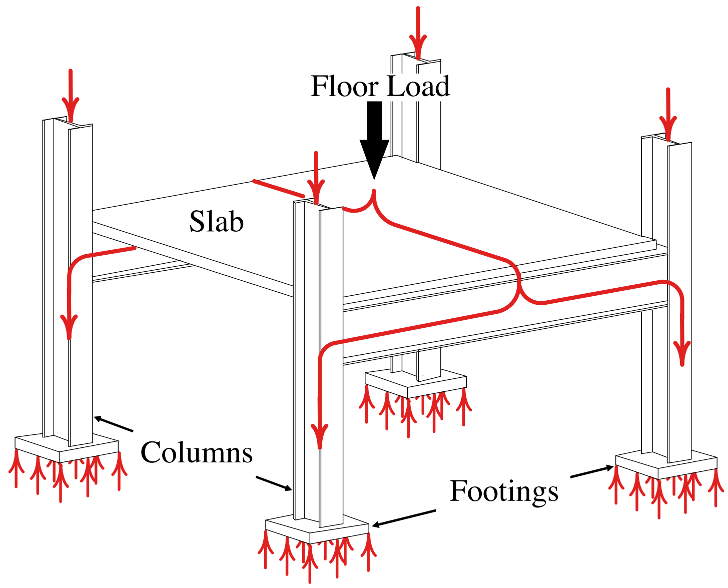


Figure 6.2.4 Load paths

6.3 Trusses

Key Questions

- What are simple trusses and how do they differ from other structural systems?
- What are the benefits and dangers of simple trusses?
- How can we determine the forces acting within simple truss systems?
- For a truss in equilibrium, why is every individual member, joint, and section cut from the truss also in equilibrium?
- How do we identify zero-force members in a truss and use their presence to simplify the analysis?

6.3.1 Introduction

A **truss** is a rigid engineering structure made up of long, slender members connected at their ends. Trusses are commonly used to span large distances with a strong, lightweight structure. Some familiar applications of trusses are bridges, roof structures, and pylons. **Planar trusses** are two-dimension trusses built out of triangular subunits, while **space trusses** are three-dimensional, and the basic unit is a tetrahedron.

In this section we will analyze a simplified approximation of a planar truss, called a **simple truss** and determine the forces the members individually support when the truss supports a load. Two different approaches will be presented: the **method of sections**, and the **method of joints**.

6.3.2 Simple Trusses

Truss members are connected to each other rigidly, by welding or joining the ends with a gusset plate. This makes the connecting joints rigid, but also make the truss difficult to analyze. To reduce the mathematical complexity in this text we will only consider **simple trusses**, which are a simplification appropriate for preliminary analysis.

Figure 6.3.1 Truss with riveted gusset plates.

Simple trusses are made of all two-force members and all joints are modeled as frictionless pins. All applied and reaction forces are applied only to these joints. Simple trusses, by their nature, are statically determinate, having a sufficient number of equations to solve for all unknowns values. While the members of real-life trusses stretch and compress under load, we will continue to assume that all bodies we encounter are rigid.

Simple trusses are made of triangles, which makes them rigid when removed from supports. Simple trusses are determinate, having a balance of equations and unknowns, following the equation:

$$\underbrace{2 \times (\text{number of joints})}_{\text{system equations}} = \underbrace{(\text{number of reaction forces}) + (\text{number of members})}_{\text{system unknowns}}$$

Commonly, rigid trusses have only three reaction forces, resulting in the equation:

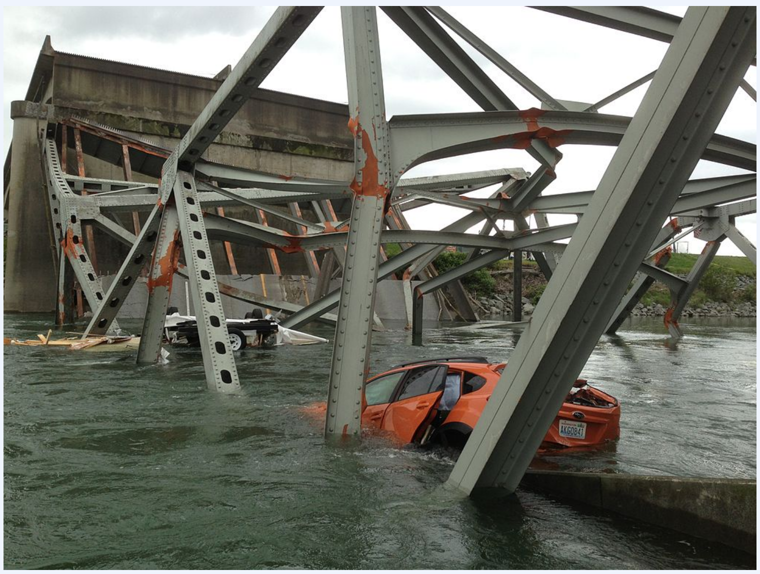
$$2 \times (\text{number of joints}) = 3 + (\text{number of members})$$

Unstable trusses lack the structural members to maintain their rigidity when removed from their supports. They can also be recognized using the equation above having more system equations on the left side of the equation above then system unknowns on the right.

Truss systems with redundant members have fewer system equations on the left side of the equation above than the system unknowns on the right. While they are indeterminate in statics, in later courses you will learn to solve these trusses too, by taking into account the deformations of the truss members.

Thinking Deeper 6.3.2 The Danger of Simple Trusses.

Simple trusses have no structural redundancy, which makes them easy to solve using the techniques of this chapter, however this simplicity also has a dark side.



These trusses are sometimes called *fracture critical trusses* because the failure of a single component can lead to catastrophic failure of the entire structure. With no redundancy, there is no alternative load path for the forces that normally would be supported by that member. You can visualize the fracture critical nature of simple trusses by thinking about a triangle with pinned corners. If one side of a triangle fails, the other two sides lose their support and will collapse. In a full truss made of only triangles, the collapse of one triangle starts a chain reaction which causes others to collapse as well.

While fracture critical bridges are being replaced by more robust designs, there are still thousands in service across the United States. To read more about two specific fracture critical collapses search the internet for the *Silver Bridge* collapse, or the *I-5 Skagit River Bridge* collapse.

6.3.3 Solving Trusses

"Solving" a truss means identifying and determining the unknown forces carried by the members of the truss when supporting the assumed load. Because trusses contains only two-force members, these internal forces are all purely axial. Internal forces in frames and machines will additionally include traverse forces and bending moments, as you will see in Chapter **??**.

Determining the internal forces is only the first step of a thorough analysis of a truss structure. Later steps would include refining the initial analysis by considering other load conditions, accounting for the weight of the members, relaxing the requirement that the members be connected with frictionless pins, and ultimately determining the stresses in the structural members and the dimensions required in order to prevent failure.

Two strategies to solve trusses will be covered in the following sections: the Method of Joints and the Method of Sections. Either method may be used, but the Method of Joints is usually easier when finding the forces in all the members, while the Method of Sections is a more efficient way to solve for specific members without solving the entire truss. It's also possible to mix and match methods.

The initial steps to solve a truss are the same for both methods. First, ensure that the structure can be modeled as a simple truss, then draw and label a sketch of the entire truss. Each joint should be labeled with a letter, and the members will be identified by their endpoints, so member AB is the member between joints A and B. This will help you keep everything organized and consistent in later analysis. Then, treat the entire truss as a rigid body and solve for the external reactions using the methods of Chapter 5. If the truss is cantilevered and unsupported at one end you may not actually need the reaction forces and may skip this step. The reaction forces can be used later to check your work.

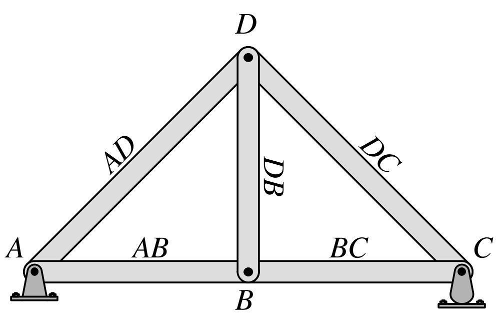


Figure 6.3.3 Truss Labels.

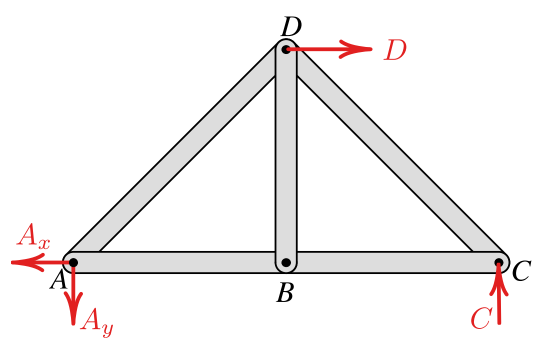
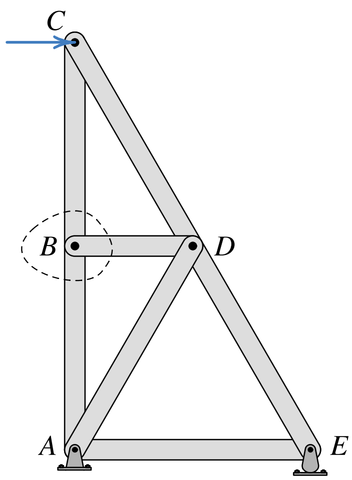


Figure 6.3.4 Free body diagram.

6.3.4 Zero-Force Members

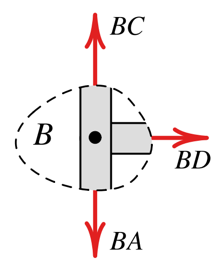
Sometimes a truss will contain one or more zero-force members. As the name implies, zero-force members carry no force and thus support no load. Zero-force members will be found when you apply equilibrium equations to the joints, but you can save some work if you can spot and eliminate them before you begin. Fortunately, zero-force members can easily be identified by inspection with two rules.

- Rule 1: If *two* non-collinear members meet at an unloaded joint, then both are zero-force members.
- Rule 2: If *three* forces (interaction, reaction, or applied forces) meet at a joint and two are collinear, then the third is a zero-force member.

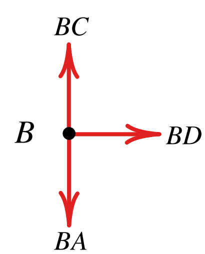


Consider the truss to the left. Assume that the dimensions, angles and the magnitude of force C are given. At joint B, there are two vertical collinear members as well as a third member which is horizontal, so Rule 2 should apply.
What does Rule 2 say about member BD? Can it tell us anything about member DA?

Cutting the members at the dotted boundary line exposes internal forces BC, BD and BA. These forces act along the axis of the corresponding members by the nature of two-force members, and for convenience have been assumed in tension although that may turn out to be incorrect.
Rule 2 applies here since BA and BC are collinear and BD is not.



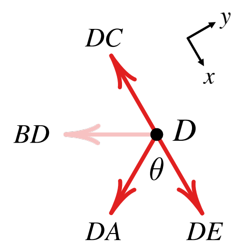
The free-body diagram of joint B may be drawn by eliminating the cut members and only showing the forces themselves. The situation is simple enough to apply the equilibrium equations in your head.



Vertically, forces BC and BA must be equal, and horizontally, force BD must be zero to satisfy $\Sigma F_x = 0$. We learn that member BD is a zero-force member.

While it is probably easiest to think about Rule 2 when the third member is perpendicular to the collinear pair, it doesn't have to be. Any perpendicular component must be zero which implies that the corresponding member is zero-force.

Finding zero-force members is an iterative process. If you determine that a member is zero-force, eliminate it and you may find others. Continuing the analysis at joint D draw its free-body diagram. Keep in mind that if one end of a member is zero-force the whole member is zero-force. Since member BD is zero-force, horizontal force BD acting on joint D is zero and need not be included on the free-body diagram, and the remaining three forces match the conditions to apply Rule 2.

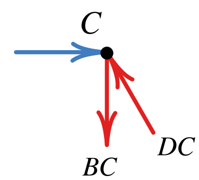


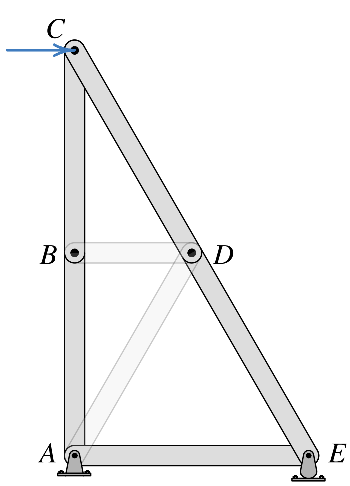
Analyzing the joint as before, but with a coordinate system aligned with the collinear pair,

$$\Sigma F_y = 0$$
$$DA \sin\theta = 0$$

This equation will be satisfied if $DA = 0$ or if $\sin\theta = 0$ but the second condition is only true when $\theta = 0°$ or $\theta = 180°$, which is not the case here. Therefore, force DA must be zero, and we can conclude that member DA is a zero-force member as well.

Finally consider joint C and draw its free-body diagram. Does either Rule apply to this joint? No. You will need to solve two equilibrium equations with this free-body diagram to find the magnitudes of forces CD and CB.
On the other hand, if the horizontal load C was not present or if either BC or DC was zero-force, then Rule 1 *would* apply and the remaining members would also be zero-force.





The final truss after eliminating the zero-force members is shown to the left. Forces $BC = BA$ and $DC = DE$ and the members may be replaced with longer members AC and CE.
The original truss has been reduced to a simpler triangular structure with only three internal forces to be found. Once you are able to spot zero force members, this simplification can be made without drawing any diagrams or performing any calculations.

Thinking Deeper 6.3.5 Why include Zero-Force Members?

You may be wondering what is the point of including a member in a truss if it supports no load. In our simplified example problems, they really are unnecessary, but in the real world, zero-force members are important for several reasons:

- We have assumed that all members have negligible weight or if not, applied half the weight to each pin. The *actual* weight of *real* members invalidates the two-force body assumption and leads to errors. Consider a vertical member -- the internal forces must at least support the member's weight.

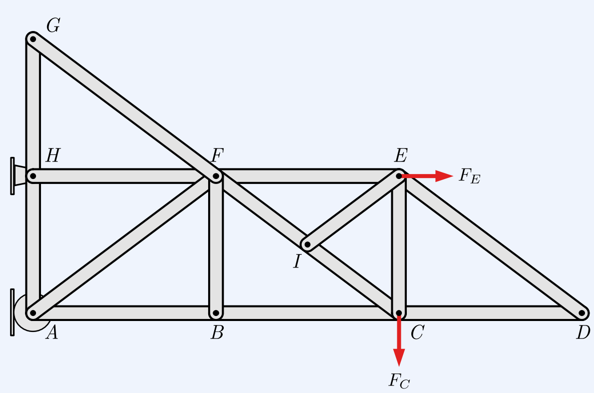
- Truss members are not actually rigid, and long slender members under compression will *buckle* and collapse. The so-called zero-force member will be engaged to prevent this buckling. In the previous example, members CD and DE are under compression and form an unstable equilibrium and would definitely buckle at pin D if they were not replaced with a single member CE with sufficient rigidity.

- Trusses are often used over a wide array of loading conditions. While a member may be zero-force for one loading condition, it will likely be engaged under a different condition — think about how the load on a bridge shifts as a heavy truck drives across.

So finding a zero-force member in a determinate truss does not mean you can discard the member. Zero-force members can be thought of as removed from the analysis, but only for the loading you are currently analyzing. After removing zero-force members, you are left with the simplest truss which connects the reaction and applied forces with triangles. If you misinterpret the rules you may over-eliminate members and be left with missing legs of triangles or 'floating' forces that have no load path to the foundation.

Example 6.3.6 Zero-Force Member Example.

Given the truss shown, eliminate all the zero-force members, and draw the remaining truss.



Solution. Rule 1:

- Due to two members meeting at unloaded joint G, both members

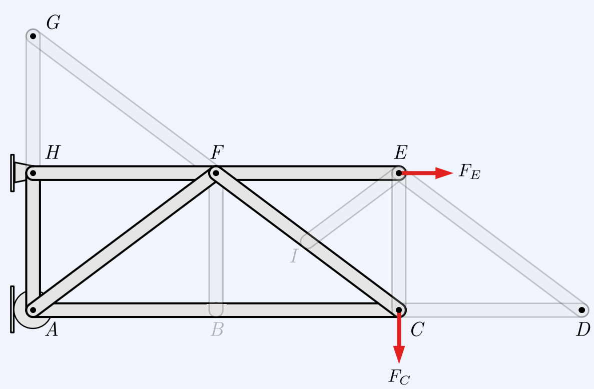
GH and FG are zero-force members

- Due to two members meeting at unloaded joint D, both members DE and CD are zero-force members

Rule 2:

- Due to three forces meeting at joining B, with two being collinear (internal forces in AB and BC) then BF is a zero-force member.
- Due to three forces meeting at joint I, with two being collinear (internal forces in IF and CI), then EI is a zero-force member. Note that member EI does not need to be perpendicular to the collinear members to be a zero force.
- After removing zero-force members EI and DE, three forces reamain at E, with two being collinear (internal force in EF and external load F_E), making EC a zero-force member.

The remaining truss is shown. Note that once EI and BF are eliminated, you can effectively eliminate the joints B and I as the member forces in the collinear members will be equal. Also notice that the truss is still formed of triangles which fully support all of the applied forces.



Try to find all the zero-force members in the truss in the interactive diagram below, once you believe you have found all of them, check out the step-by-step solution in the interactive.

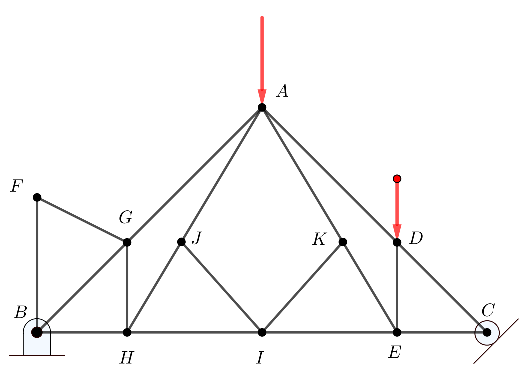


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Figure 6.3.7 Identify zero-force members.

6.4 Method of Joints

Key Questions

- What are the important components to include on a free-body diagram of a joint in a truss?
- How are the solutions found at one joint used to create an accurate free-body diagram of another joint?
- How do we ensure that tension or compression in a member is properly represented?

The **method of joints** is a process used to solve for the unknown forces acting on members of a truss. The method centers on the joints or connection points between the members, and it is most useful when you need to solve for all the unknown forces in a truss structure.

The joints are treated as particles subjected to force by the connected members and any applied loads. As the joints are in equilibrium and the forces are concurrent, $\Sigma\mathbf{F} = 0$ can be applied, but the $\Sigma M = 0$ equation provides no information.

For planar trusses, each joint yields two scalar equations, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, and so two unknowns can be found. Therefore, a joint can be solved when there are one or two unknowns forces and at least one known force acting on it.

Forces are transferred from joint to joint by the connecting members, so when unknown forces on a joint are found, the corresponding forces on adjacent joints are also found.

6.4.1 Procedure

The procedure is straightforward application of rigid body and particle equilibrium

1. Determine if the structure is a truss and if it is determinate. See Subsection 6.3.2
2. Identify and remove all zero-force members. This is not required, but will eliminate unnecessary computations. See Subsection 6.3.4.
3. Determine if you need to find the external reactions. If you can identify a solvable joint immediately, then you do not need to find the external reactions.

   A solvable joint includes one or more known forces and no more than two unknown forces. If there are no joints that satisfy this condition then you will need to find the external reactions before proceeding, using a free-body diagram of the entire truss.

4. Identify a solvable joint and solve it using the methods of Chapter 3. When drawing free-body diagrams of joints you should

    - Represent the joint as a dot.
    - Draw all known forces in their known directions with arrowheads indicating their sense. Known forces are the given loads, and forces determined from previously solved joints.
    - Assume the sense of unknown forces. A common practice is to assume that all unknown forces are in tension, i.e. pulling away from the free-body diagram of the pin, and label them based on the member they represent.

    Finally, write out and solve the force equilibrium equations for the joint. If you assumed that all forces were tensile earlier, negative answers indicate compression.

5. Once the unknown forces acting on a joint are determined, carry these values to the adjacent joints and repeat step four until all the joints have been solved. Take care when transferring forces to adjoining joints to maintain their sense — either tension or compression.

6. If you solved for the reactions in step two, you will have more equations available than unknown forces when you reach the last joint. The extra equations can be used to check your work.

Rather than solving the joints sequentially, you could write out the equations for all the joints first and solve them simultaneously using a matrix solution, but only if you have a computer available as large matrices are not typically solvable with a calculator.

The interactive below shows a triangular truss supported by a pin at A and a roller at B, and loaded at joint C. You can see how the reactions and internal forces adjust as you vary the load at C. You can solve it by starting at joint C and solving for BC and CD, then moving to joint B and solving for AB Joint A can be used to check your work.

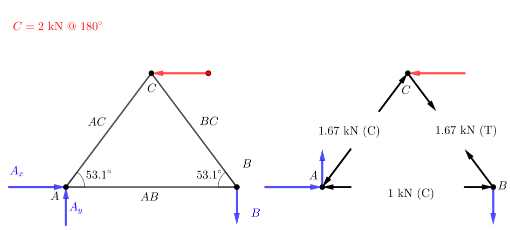

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Figure 6.4.1 Internal and external forces of a simple truss.

6.5 Method of Sections

Key Questions

- How do we determine an appropriate section to cut through a truss?
- How are equilibrium equations applied to a section?

The **method of sections** is used to solve for the unknown forces within specific members of a truss without solving for them all. The method involves dividing the truss into sections by cutting through the selected members and analyzing the section as a rigid body. The advantage of the Method of Sections is that the only internal member forces exposed are those which you have cut through, the remaining internal forces are not exposed and thus ignored.

6.5.1 Procedure

The procedure to solve for unknown forces using the method of sections is

1. Determine if a truss can be modeled as a simple truss.
2. Identify and eliminate all zero-force members. Removing zero-force members is not required but may eliminate unnecessary computations.
3. Solve for the external reactions, if necessary. Reactions will be necessary if the reaction forces act on the section of the truss you choose to solve below.
4. Use your imaginary chain saw to cut the truss into two pieces by cutting through some or all of the members you are interested in. The cut does not need to be a straight line.

   Every cut member exposes an unknown internal force, so if you cut three members you'll expose three unknowns. Exposing more than three members is not advised because you create more unknowns than available equilibrium equations.
5. Select the easier of the two halves of the truss and draw its free-body diagram.
   - Include all applied and reaction forces acting on the section, and show known forces acting in their known directions.
   - Draw unknown forces in assumed directions and label them. A common practice is to assume that all unknown forces are in tension and label them based on the endpoints of the member they represent.

6. Write out and solve the equilibrium equations for your chosen section. If you assumed that unknown forces were tensile, negative answers indicate compression.

7. If you have not found all the required forces with one section cut, repeat the process using another imaginary cut or proceed with the method of joints if it is more convenient.

The interactive below demonstrates the method of sections. The internal forces in the truss members are exposed by cutting through the truss at three locations. The known loads are shown in red, and the unknown reactions F_x, A_x and A_y, and unknown member forces are shown in blue. All members are assumed to be in tension. In this situation, it is not necessary to find the reactions if the section to the right of the cut is selected.

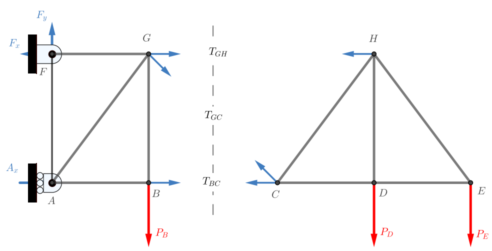

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Figure 6.5.1 Method of sections demonstration.

6.6 Frames and Machines

Key Questions

- How are frames and machines different from trusses?
- Why can the method of joints and method of sections not be used for frames and machines?
- How do we identify if a structure is independently rigid?
- How do we apply equilibrium equations to each member of the structure, and ensure that the sense of a force appearing on multiple free-body diagrams is consistent?

Frame and machines are engineering structures that contain at least one multi-force member. As their name implies, multi-force members have more than two concentrated loads, distributed loads, and/or couples applied to them and therefore are not two-force members. Note that all bodies we investigated in Chapter 5 were all multi-force bodies.

Frames are rigid, stationary structures designed to support loads and must include at least one multi-force member.

Machines are non-rigid structures where the parts can move relative to one another. Generally they have an input and an output force and are designed produce a mechanical advantage. Note that all machines in this text are in static equilibrium by their interacting and applied forces.

Though there is a design difference between frames and machines they are grouped together because they can both be analyzed using the same process, which is the subject of this section.

Figure 6.6.1 Frames are rigid objects containing multi-force members.

Figure 6.6.2 Machines contain multi-force members that can move relative to one another.

6.6.1 Analyzing Frames and Machines

Analyzing a frame or machine means determining all applied, reaction, and internal forces and couples acting on the structure and all of its parts.

Multi-part structures are analyzed by mentally taking them apart and analyzing the entire structure and each part separately. Each component is analyzed as an separate rigid body using the techniques we have already seen.

Although we can separate the structure into parts, the parts are not independent since, by Newton's Third Law, every interaction is half of a complementary pair. For every force or moment of body A on B there is an equal-and-opposite force or moment of body B on body A and the free-body diagrams must reflect this. Incorrect representation of these interacting pairs on free-body diagrams is a common source of student errors.

Once the frame or machine is disassembled and free-body diagrams have been drawn, the structure is analyzed by applying equations of equilibrium to free-body diagrams, exactly as you have done before —there's nothing new here.

The difficulty arises first in selecting objects and drawing correct free-body diagrams and second, in identifying an efficient solution strategy since you usually won't be able to completely solve a diagram without first finding the value of an unknown force from another diagram.

In Chapter 5 we saw that each two-dimensional free-body diagram results in up to three linearly independent equations. By disassembling the structure we now have more free-body diagrams available, and can use them to find more unknown values. Here's a few more details on the number of equations that come from each type of two-dimensional free-body diagram:

- *Two-force members.*

  One equation. Two-force members can be recognized as either a cable or a weightless link with all forces coming from two frictionless pins. The force at one pin is equal and opposite to the force on the other placing the body in tension or compression.

- *Objects with concurrent forces and no couple-moments.*

  Two equations. These are the problems you solved in Chapter 3. There are two equations available $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

- *Multi-force rigid body with non-concurrent forces and/or couples.*

  Three equations. These are the most general body types. Use $\Sigma F_x = 0$, $\Sigma F_y = 0$, and$\Sigma M = 0$ to solve for three unknowns.

Procedure

The process used to analyze frames and machines is outlined below

1. Determine if the entire structure is independently rigid. An independently rigid structure will hold it shape even when separated from its supports. Look for triangles formed among the members, as triangles are inherently rigid. If it is not independently rigid, the structure will collapse when the supports are removed.

   If the structure is not independently rigid, skip to the next step. Otherwise, model it as a single rigid body and determine the external reaction forces.

2. Draw a free-body diagram for each of the members in the structure. You must represent all forces acting on each member, including:

   - Applied forces and couples and the weights of the components if non-negligible.
   - Interaction forces due to two-force members. There will be force of unknown magnitude but the known direction at points connected to two-force members. The forces will act along the line between the two connection points.
   - All reaction forces and moments at the connection points between members. Forces with an unknown magnitude and direction are usually represented by unknown x and y components, but can also be represented as a force with unknown magnitude acting in an unknown direction.

All interaction forces and moments between connected bodies *must* be shown as equal-and-opposite action-reaction pairs.

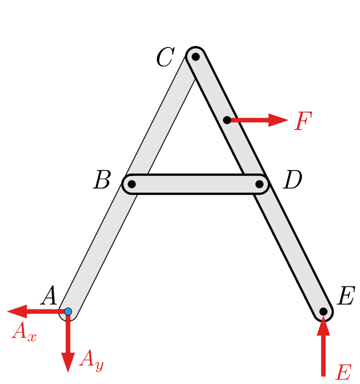


Figure 6.6.3 Free-body diagram of a rigid frame with pin at A, roller at E, and load at F.

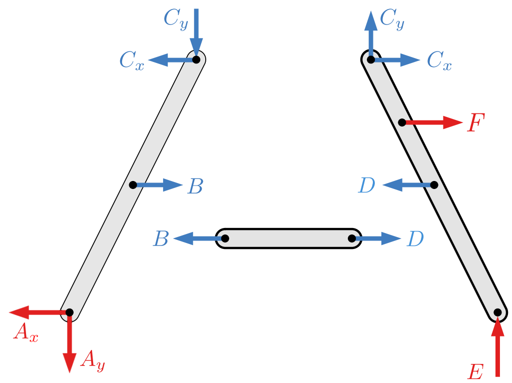


Figure 6.6.4 Free body diagrams of the individual components. External forces are red, exposed action-reaction pairs in blue.

3. Write out the equilibrium equations for each free-body diagram.

4. Solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. Negative magnitudes indicate that the assumed direction of that term was incorrect, and the actual force/moment is opposite the assumed direction.

In the following example, we'll discuss how to select objects, distinguish external and internal loads, draw consistent and correct free-body diagrams, and identify a good solution strategy.

Free-body diagram of structures

Drawing free-body diagrams of complex frames and machines can be tricky. In this section we will walk through the process of selecting appropriate objects and drawing consistent and correct free-body diagrams in order to solve a typical machine problem.

The toggle clamp shown in Figure 6.6.5 is used to quickly secure wooden furniture parts to the bedplate of a CNC router in order to cut mortise and tenon joints. The component parts are shown and named in Figure 6.6.6.

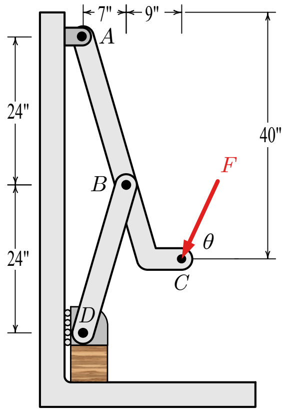


Figure 6.6.5 Original diagram

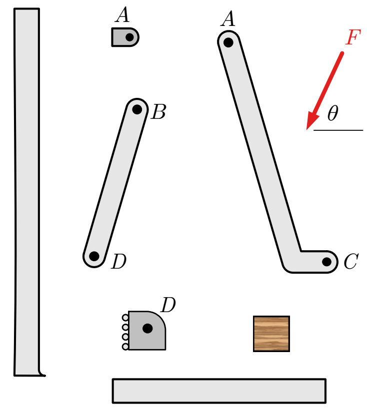


Figure 6.6.6 Component parts.

This original diagram is not a free-body diagram because all the forces necessary to hold the objects in equilibrium are not indicated. The only force shown is F, which is supplied by some external agent, presumably the machine operator. We must assume that the wall and floor are still attached to the world and held fixed.

To perform an equilibrium analysis, we need to develop one or more free-body diagrams and apply the equations of equilibrium to them. Free-body diagrams can be drawn for the entire structure, each individual part, and for any combination of connected parts. Not all these diagrams will be needed however, and part of the challenge of solving these problems is selecting and drawing only the ones you need. In any event, a clear decision must always be made about what is part of the free-body and what is not.

When we separate one body from another loads will appear on both bodies which act to constrain them as they were constrained before the separation. These forces and moments must be represented on the free-body diagrams consistently as halves of equal-and-opposite action-reaction pairs.

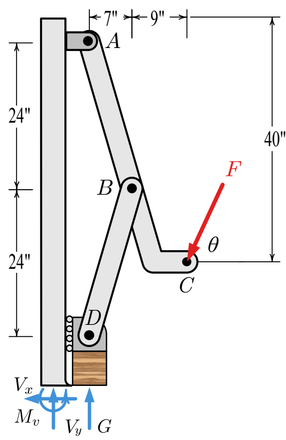
For this discussion we will progressively exclude parts from the original structure and draw the free-body diagram of what remains. In so doing we will clarify the difference between internal and external forces, recognize and take advantage of two-force bodies, and provide some tips for drawing correct free-body diagrams. In an actual situation you will not need to draw all these diagrams, instead you should think through the situation and draw only the diagrams you will need for a solution.

It is helpful to consider which loads are known and which are unknown as you prepare free-body diagrams. In planar problems a free-body diagram with three or fewer unknowns may be solved immediately. When there are more than three unknowns, you must incorporate information from other diagrams to complete the solution.

Exclude the floor. To begin, we can remove the floor from the system. Everything except the floor is now included as our body; only the floor is excluded.

The floor was in contact with the other objects at the ground and also at the connection between the floor and the wall.

Since we don't know how the wall and the floor are connected we will assume they were fixed together. We also have to model how the wall is attached to the rest of the world. The fixed support from wall-to-world and wall-to-floor can be combined to be a single set of three loads which we represent as horizontal and vertical forces V_x and V_y, and a concentrated moment M_v.



Included	Excluded
Lever ABC, Short Link BD, Wooden Block, Roller D, Wall, Bearing A	Floor

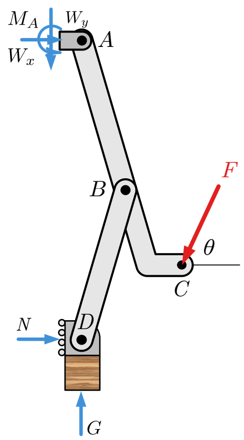
Figure 6.6.7 Free-body Diagram 1

The effect of the floor on the block is represented by a single vertical force G which holds the block the same way the floor was previously supporting it; the loads you add must constrain your object the same way they were constrained in the real world. This representation is really a simplification of the actual situation since the force of the floor is really distributed over the bottom surface of the block; however, this simplification is justified in much the same way as we represent the weight of an object as a single force acting at its center of gravity.

Tips.

- Include friction if it's given or obvious.
- Internal forces in rigid bodies should be modeled as a fixed support.
- If you need info which you don't have, select a variable to act as its name.

Exclude the wall. If you next remove the wall, forces G and F remain from before, but we now expose four loads from where the wall was connected to what is now our body; a normal force N at the roller and three loads from the fixed support between the bearing block and the wall W_x, W_y, and M.



Included	Excluded
Lever ABC, Short Link BD, Wooden Block, Roller D, Bearing A	Floor, Wall

Figure 6.6.8 Free-body diagram 2

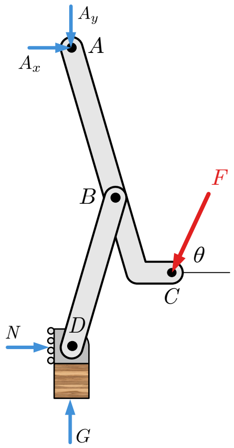
Note that the reactions between the wall and the floor are no longer included in the free-body diagram because they are both on the same side of the included-excluded table. Only loads that cross from included to excluded produce a load on the free-body diagram.

Tips.

- Every force needs a point of application and a clear arrowhead.
- Indicate any distances and angles needed and not available on the original diagram.
- Define the direction of forces which are not vertical or horizontal with an angle from a reference direction.
- Define a coordinate system unless you are using the standard x-y axes.
- Do not add forces that don't act on your body.

Exclude the bearing at A. We are not interested in the loads between the bearing block and the wall W_x, W_y, and M and further, the free-body diagram still includes too many unknowns to solve.

After removing the bearing we reduce the unknowns at A to two because the bearing block and the lever are connected with a pin while the bearing block and wall were connected with a fixed support. The loads W_x, W_y, and M and V_x, V_y, and M_v are not included on this free-body diagram because they don't act on this object.



Included	**Excluded**
Lever ABC, Short Link BD, Wooden Block, Roller D,	Floor, Wall, Bearing A

Figure 6.6.9 Free-body diagram 3

The load from the short link at B is does not appear on this free-body diagram because it is internal. Internal loads connect two parts of the body together. They should not be included in the free-body diagram because they always occur in equal and opposite pairs which cancel each other out.

Tips.

- Look for free-body diagrams which include only three unknowns in two dimensions or six unknowns in three.
- Don't include internal loads on your free-body diagrams.

Examine the wooden block.

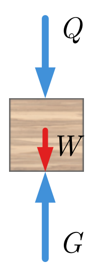


Figure 6.6.10 Free-body diagram 4 (block)

A free-body diagram of the block shows the clamping which we are seeking.
Note that $Q \neq G$. These forces are given different nan they may have different magnitudes. If the weight of the small[1] in comparison to the other forces acting on the may be neglected, in which case $Q = G$ and they could the same name.

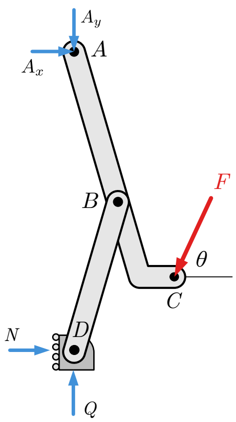
Tips.

- If the two forces are not the same don't identify them by the same name.
- Make as few assumptions as you possibly can. Make a note of any assumptions you make.
- In textbook problems, if the weight of an object is not mentioned it

[1](less than about 0.1%)

may be neglected.

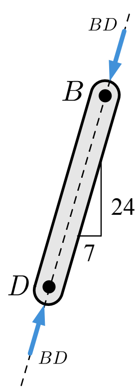
Exclude the wooden block. We can further simplify the diagram by removing the wooden block, leaving only the roller, short link and lever.



Included	**Excluded**
Lever ABC, Short Link BD, Roller D	Floor, Wall, Bearing A, Wooden Block

Figure 6.6.11 Free-body diagram 5 (lever and link)

Examine the short link BD. The short link BD is a two-force body and as discussed in Subsection 3.3.3 can only be in equilibrium if the forces at B and D are equal-and-opposite and act along a line passing through these two points. This means that the 24:7 slope of the link determines the direction of force BD.



When drawing free-body diagrams, forces with known directions should be drawn pointing in that direction rather than breaking them into components, otherwise you may lose track of the fact that the x and y components are not independent but are actually related by the direction of the force.

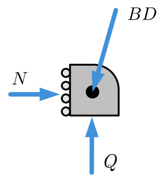
Figure 6.6.12 Free-body diagram 6 (short link)

Here we have assumed that the forces acting on the link are compressive. If the equilibrium equations produce a positive value for BD our assumption is proved correct, while a negative result indicates that we were wrong and the link is actually in tension.

Tips.

- A short-link is a two-force body.

- Recognize two-force bodies because they give you information about direction.
- Represent the force of a two-force bodies as a force with unknown magnitude acting along a known line of action, not as x and y components.
- If you don't know the sense of a force along its line of action, assume one. If you guess wrong, the analysis will give you a negative value.



Note that the force BD acting on the roller is shown poi down and to the left. This is the opposite to the force actin the link at D, which acts up and to the right. These two f act in opposite directions because they are an action-reac pair.

Figure 6.6.13 Free-body diagram 7 (roller)

Examine the roller at D.

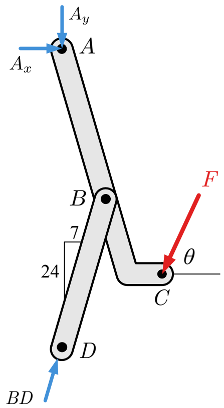
The roller is a three-force body, so the lines of action of N, Q, and BD are coincident and it may be treated as a particle. Equilibrium analysis shows that N and Q must oppose the horizontal and vertical components of force BD.

The clamping force Q produced by the toggle clamp appears on this free-body diagram so it will be important later for the solution.

Tips.

- Recognize three-force bodies and use their special properties to your advantage.
- Use the same name for the exposed forces on interacting bodies since they are equal-and-opposite halves of an action-reaction pair.

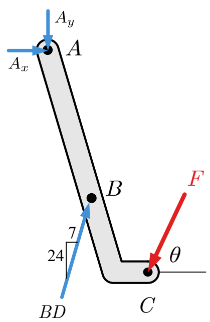
Exclude the roller. We can further simplify the free-body diagram by removing the roller. The roller and short link are connected with a pin but, for equilibrium, the forces acting on a short link (or any two-force body) must share the same line of action — the line connecting its endpoints; otherwise, components perpendicular to this line would produce an unbalanced moment about the other endpoint.



Included	**Excluded**
Lever ABC, Short Link BD	Floor, Wall, Bearing A, Wooden Block, Roller D

Figure 6.6.14 Free-body diagram 8

Exclude the short link. The previous free-body diagram has three unknowns and can be solved but the free-body diagram of the lever by itself is also correct, and this is the free-body diagram that most people begin with.



Included	**Excluded**
Lever ABC	Floor, Wall, Bearing A, Wooden Block, Roller D, Short Link BD

Figure 6.6.15 Free-body diagram 9 (lever)

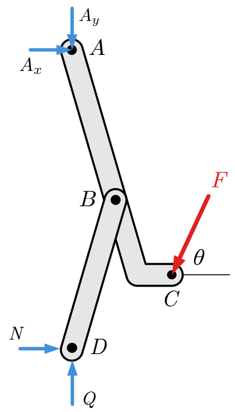
The load BD acting on the lever in this diagram has the same magnitude, direction, and line of action as the load acting on the short link at D, so this can be thought of as sliding a force along its line of action — an equivalent transformation.

The following loads are not shown here because they act between two objects which are not part of the body:

- the loads between the bearing block and the wall W_x, W_y, and M,
- the loads between the floor and the wall V_x, V_y, and M_v,
- the load between the block and the floor G, and
- the load between the roller and the wall N.

All of the free-body diagrams we have drawn are correct, though not all are necessary. Generally we only draw the free-body diagrams needed for the solution. These diagrams form a chain which connect the known input forces to the desired output forces. When solving frames and machines, think carefully about what you know and what you need to solve for: that determines which free-body diagrams you will need. Taking a few moments to consider what unknowns you'd have at each step can help you optimize your problem-solving effort.

You should recognize that it is possible to draw incorrect free-body diagrams which produce correct results. Consider the diagram below.



This diagram doesn't accurately represent what is happening at pin D.

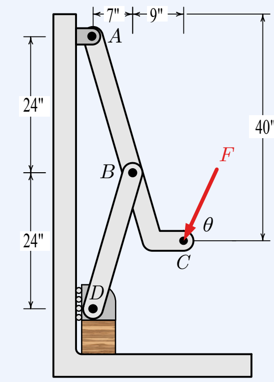
Figure 6.6.16 Free-body diagram 10 (Subtly incorrect)

Forces N and Q do not actually act on the short link at D. Force N acts between the roller and the wall and clearly this diagram doesn't include the roller. Similarly Q acts between the block and the roller. These forces don't belong on the free-body diagram even though they are equal to the x and y components of force BD. Only forces which cross the imaginary boundary between the object and the rest of the world belong on the free-body diagram.

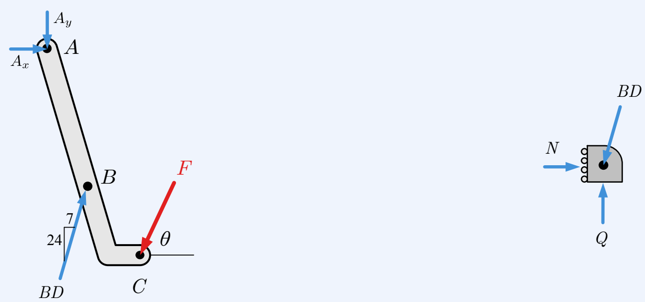
Students are inclined argue that this free-body diagram is statically equivalent to Figure 6.6.11 and it produces the correct answer so it must be OK. It isn't correct because it reflects a misunderstanding about what you are modeling and what you aren't. Other engineers using your FBDs need to know what you are modeling. The FBD is the key to your analysis of the real world.

Example 6.6.17 Toggle Clamp.

Knowing that angle $\theta = 60°$, find the vertical clamping force acting on the piece at D and the magnitude of the force exerted on member ABC at pin B in terms of force F applied to the clamp arm at C.



Solution. For this problem, we need two free-body diagrams. The first links the input force F to the link force BD, and the second links BD to the clamping force Q.



(a) FBD I **(b)** FBD II

Figure 6.6.18

We will assume the two-force member BD is in compression based on the physical situation. The forces acting on the link, lever and roller are all directed along a line-of-action defined by a 7-24-25 triangle. Similar triangles gives

$$BD_x = \left(\frac{7}{25}\right) BD$$
$$BD_y = \left(\frac{24}{25}\right) BD.$$

Applying $\sum M = 0$ at A to the free-body diagram of the lever gives BD in terms of F.

$$\text{FBD I:} \quad \Sigma M_A = 0$$
$$BD_x(24) + BD_y(7) - F_x(40) - F_y(16) = 0$$
$$\left(\frac{7}{25}BD\right)(24) + \left(\frac{24}{25}BD\right)(7) = (F\cos 60°)(40) + (F\sin 60°)(16)$$

$$\begin{aligned} 13.44BD &= 33.86F \\ BD &= 2.52F \end{aligned}$$

The positive sign on the answer reveals that our assumption that member BD was in compression was correct.
Applying $\sum F_y = 0$ to the free-body diagram of the roller will give Q in terms of F.

$$\begin{aligned} \text{FBD II:} \quad \Sigma F_y &= 0 \\ Q - BD_y &= 0 \\ Q &= \frac{24}{25} BD \\ &= \frac{24}{25}(2.52F) \\ &= 2.42F \end{aligned}$$

While you could certainly find A_x, A_y and N using other equilibrium equations they weren't asked for and we don't bother to find them.

Thinking Deeper 6.6.19 Why does the Method of Joints work on trusses but not on Frames or Machines?

We can solve trusses using the methods of joints and method of sections because all members of a simple truss are two-force bodies. Cutting a truss member exposes an internal force which has an unknown scalar magnitude, but a known line of action. The force acts along the axis of the member, and causes no bending if the member is straight. Cutting a truss member exposes *one* unknown.
Frames and machines are made of multi-force members and cutting these, in general, exposes:

- A force with an unknown magnitude acting in an unknown direction, and
- A bending moment at the plane of the cut.

Cutting a two-dimensional multi-force member exposes *three* unknowns, and *six* are exposed for a three-dimensional body. The number of unknowns quickly eclipses the available equations rendering the problem impossible to solve.
Bottom line: use method of sections and joints *only* for trusses made of two-force straight members; for all other multi-force rigid body systems draw and analyze free-body diagrams of the components.

6.7 Summary

The various equilibrium topics we have covered and the associated problem solving techniques are summarized below.

You should be able to recognize these situations, draw the associated free-body diagrams and solve for the unknowns of each case.

Particle Equilibrium. An object may be treated as a particle when the forces acting on it are coincident, that is, all of their lines of action intersect at a common point. In this case, they produce no moment to rotate the object, and $\Sigma \mathbf{M} = 0$ is not helpful. The applicable equation is

$$\Sigma \mathbf{F} = 0,$$

which produces two scalar equations in two dimensions and three scalar equations in three dimensions.

Rigid Body Equilibrium. A rigid body can rotate and translate so both force and moment equilibrium must be considered.

$$\Sigma \mathbf{F} = 0$$
$$\Sigma \mathbf{M} = 0$$

In two dimensions, these equations produce in two scalar force equations and one scalar moment equation. Up to three unknowns can be determined.

In three dimension, they produce three scalar force equations and scalar three moment equations. Up to six unknowns can be determined.

Trusses. A truss is a structure which consists entirely of two-force members and only carries forces at the joints connecting members. Two-force members and loading at joints allows free-body diagram of the joints to expose the axial loads in members.

In addition to the equations provided by treating the entire truss as a rigid body, each joint provides two additional equations for two-dimensional trusses, and three for non-planar trusses.

Frames and Machines. Frames and machines are structures which contain multiple rigid body systems. Frames don't move and are designed to support loads. Machines are generally designed to multiply forces, and usually have moving parts. Both frames and machines can be solved using the same methods.

All interactions between bodies are equal and opposite action-reaction pairs. When solving frames and machines

- Two-force members provide one useful equilibrium equation, and can determine one unknown.

- In two dimensions, rigid bodies result in two scalar force equations and one scalar moment equation. Up to three unknowns can be determined.
- In three dimensions, rigid bodies produce three scalar force equations and scalar three moment equations. Up to six unknowns can be determined.

6.8 Exercises (Ch. 6)

Truss: Method of Joints	**0/105**
Solve a joint	0/30 Not attempted
Cantilever truss	0/30 Not attempted
Kingpost truss	0/45 Not attempted
Truss: Method of Sections	**0/100**
Zero force members	0/20 Not attempted
Cantilever truss	0/15 Not attempted
Howe Truss	0/25 Not attempted
Pratt Truss	0/30 Not attempted
Maximum load	0/10 Not attempted
Frames	**0/140**
Crossbuck frame	0/30 Not attempted
Frame: A-Frame with load	0/30 Not attempted
Frame: A-frame Difficulty 1	0/20 Not attempted
Frame: A-frame Difficulty 2	0/20 Not attempted
Frame: A-frame Difficulty 3	0/20 Not attempted
Frame: A-frame Difficulty 4	0/20 Not attempted
Machines	**0/107**
Machine: Pliers	0/20 Not attempted
Machine: Vice grips	0/20 Not attempted
Machine: Pruning Shears	0/25 Not attempted
Machine: Excavator	0/22 Not attempted
Machine: lopping shears	0/20 Not attempted

Chapter 7

Centroids and Centers of Gravity

A **centroid** is the geometric center of a geometric object: a one-dimensional curve, a two-dimensional area or a three-dimensional volume. Centroids are useful for many situations in Statics and subsequent courses, including the analysis of distributed forces, beam bending, and shaft torsion.

Two related concepts are the **center of gravity**, which is the average location of an object's *weight*, and the **center of mass** which is the average location of an object's *mass*. In many engineering situations, the centroid, center of mass, and center of gravity are all coincident. Because of this, these three terms are often used interchangeably without regard to their precise meanings.

We consciously and subconsciously use centroids for many things in life and engineering, including:

> *Keeping your body's balance:* Try standing up with your feet together and leaning your head and hips in front of your feet. You have just moved your body's center of gravity out of line with the support of your feet.
>
> *Computing the stability of objects in motion like cars, airplanes, and boats:* By understanding how the center of gravity interacts with the accelerations caused by motion, we can compute safe speeds for sharp curves on a highway.
>
> *Designing the structural support to balance the structure's own weight and applied loadings on buildings, bridges, and dams:* We design most large infrastructure not to move. To keep it from moving, we must understand how the structure's weight, people, vehicles, wind, earth pressure, and water pressure balance with the structural supports.

You probably have already developed a good intuition about centroids and centers of gravity based upon your life experience, and can roughly estimate their location when you look at an object or diagram. In this chapter you will learn to locate them precisely using two techniques: integration **??** and the method of composite parts **??**.

7.1 Weighted Averages

You certainly know how to find the average of several numbers by adding them up and dividing by the number of values, so for example the average of the first four positive integers is

$$\frac{1+2+3+4}{4} = 2.5$$

More formally, if a is a set with n elements then the average, or ***mean,*** value is

$$\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i = \frac{a_1 + a_2 + \cdots + a_n}{n}. \tag{7.1.1}$$

This average is also called the **arithmetic mean**. When calculating an arithmetic mean, each number is equally important when evaluating the average. The overbar symbol is often used to indicate that a quantity is a mean value.

In situations where some values are more important than others, we use a **weighted average**. A familiar example is your grade point average. Your GPA is calculated by weighting your grade for each class by the credits for that class, then dividing by the total credits you have taken. The credit values are called the **weighting factors**.

In general terms a weighted average is

$$\bar{a} = \frac{\sum a_i w_i}{\sum w_i} \tag{7.1.2}$$

Where a_i are the values we are averaging and w_i are the corresponding weighting factors. The weighting factors may be different for each item being averaged, so w_i is the weighting factor for value a_i. In this book we will not write the limits on the sums, and understand that the intent is always to sum over all the values. Notice that if the weighting factors are all identical, they can be factored out of the sums so the weighted average and the arithmetic mean will be the same.

Weighted averaging is used to find centroids, centers of gravity and centers of mass, the subject of this chapter. All three are *points* located at the "center" the object, but the meaning of "center" depends on the weighting factors. Area or volume are the factors used for centroids, weight for center of gravity, and mass for center of mass.

Example 7.1.1 Course Grades.

The mechanics syllabus says that there are two exams worth 25% each, homework is 10%, and the final is worth 40%. You have a 40 on the first exam, a 80 on the second exam, and your homework grade is 90.
What do you have to earn on the final exam to get a 70 in the class?

Solution. Your known grades and the weighting factors are

$$G_i = [40, 80, 90, FE]$$

$$w_i = [25\%, 25\%, 10\%, 40\%]$$

Find final exam score FE so that your average grade $\bar{G}$ is 70%.

$$\begin{aligned}
\bar{G} &= \frac{\sum G_i w_i}{\sum w_i} \\
70 &= \frac{(40 \times 0.25) + (80 \times 0.25) + (90 \times 0.1) + (FE \times 0.4)}{(0.25 + 0.25 + 0.1 + 0.4)} \\
FE &= \frac{70(1) - (10 + 20 + 9)}{0.4} = 77.5.
\end{aligned}$$

7.2 Center of Gravity

So far in this book we have always taken the weight of an object to act at a point at its center. This is the **center of gravity**: the point where all of an object's weight may be concentrated and still have the same *external* effect on the body. In this chapter we will learn to actually locate this point.

We will indicate the center of gravity with a circle with black and white quadrants, and call its coordinates $(\bar{x}, \bar{y})$ or $(\bar{x}, \bar{y}, \bar{z})$. This point represents the average location of all the particles which make up the body.

The center of gravity of a body is fixed with respect to the body, but the coordinates depend on the choice of coordinate system. For example, in Figure 7.2.1 the center of gravity of the block is at its geometric center meaning that $\bar{x}$ and $\bar{y}$ are positive, but if the block is moved to the left of the y axis, or the coordinate system is translated to the right of the block, $\bar{x}$ would then become negative.

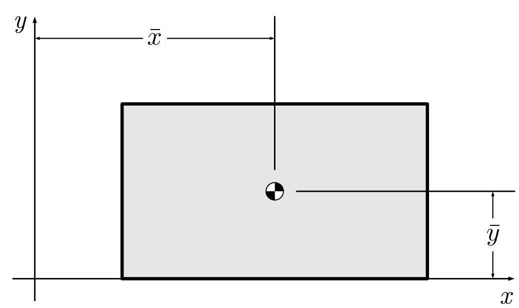


Figure 7.2.1 Location of the centroid, measured from the origin.

Lets explore the center of gravity of a familiar object. Take a pencil and try to balance it on your finger. How do you decide where to place it? You likely supported it roughly in the middle, then adjusted it until it balanced. You found the point where the moments of the weights on either side of your finger were in equilibrium.

Let's develop this balanced moment idea mathematically.

Assume that the two halves of the pencil have known weights acting at points 1 and 2. How could we replace the two weights with a single statically equivalent force? Recall from Section 4.8 that statically equivalent systems produce the same external effect on the object —the net force on the object, and the net moment about any point don't change. An upward force at this point will support the pencil without tipping.

To be equivalent, the total weight must equal the total weight of the parts. $W = W_1 + W_2$. Common sense also tells us that W will act somewhere between W_1 and W_2.

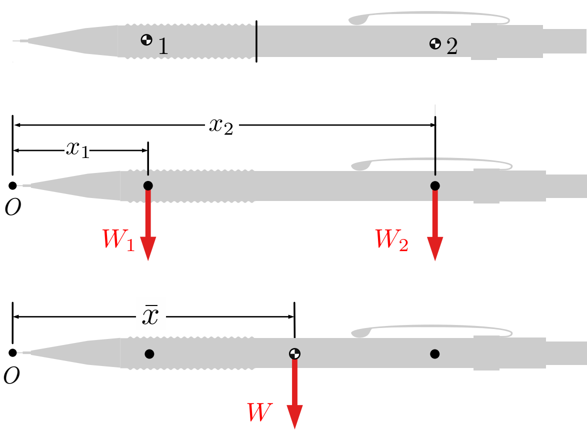


Figure 7.2.2 (top) Side view of a pencil representing each half as a particle. (middle) A force diagram showing the weights of the two particles. (bottom) An equivalent system consisting of a single weight acting at the pencil's center of gravity.

Next, let's do the mathematical equivalent of sliding your finger back and forth until a balance point is located. Pick any point O to be the origin, then calculate the total moment about O due to the two weights.

The sum of moments around point O can be written as:

$$\sum M_O = -x_1 W_1 - x_2 W_2$$

Notice that the moment of both forces are clockwise around point O, so the signs are negative according to the right-hand rule. We want a single equivalent force acting at the (unknown) center of gravity. Call the distance from the origin to the center of gravity $\bar{x}$.

$\bar{x}$ represents the mean distance of the weight, mass, or area depending on the context of the problem. We are evaluating weights in this problem, so $\bar{x}$ represents the distance from O to the center of gravity.

The sum of moments around point O for the equivalent system can be written as:

$$\sum M_O = -\bar{x} W$$

The moment of total weight W is also clockwise around point O, so the sign of moment will also be negative according to the right-hand rule. Since the two representation are equivalent we can equate them and solve for $\bar{x}$.

$$-\bar{x} W = -x_1 W_1 - x_2 W_2$$
$$\bar{x} = \frac{x_1 W_1 + x_2 W_2}{W_1 + W_2}$$

This result is exactly in the form of (7.1.2) where the value being averaged is distance x and the weighting factor is the weight of part W_i and the result is the mean distance $\bar{x}$.

The pencil was made up of two halves, but this equation can easily be extended n discrete parts. The resulting general definition of the centroidal coordinate $\bar{x}$ is:

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \tag{7.2.1}$$

where:

W_i is the weight of part i,

$\bar{x}_i$ is the x coordinate of the center of gravity of element i, and

$\sum$ is understood to mean "sum all parts" so there is no need to write $\sum\limits_{i=1}^{n}$.

The numerator is the **first moment** of force which is literally a moment of force as we defined it in Chapter 3. The denominator is the sum of the weights of the pieces, which is the weight of the whole object. We will soon also see first moments of mass and first moments of area and in Chapter **??**, we will introduce **second moments**, which are the integral of some quantity like area, multiplied by a distance *squared*.

We treated the pencil as a one-dimensional object, so this discussion focused on $\bar{x}$. There are similar formula for the other dimensions as well

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \quad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \quad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}. \tag{7.2.2}$$

In words, these equations say

$$\text{distance to CG} = \frac{\text{sum of first moments of weight}}{\text{total weight}}$$

They apply to any object which can be divided into discrete parts, and they produce the coordinates of the object's center of gravity.

Question 7.2.3

Can you explain why the center of gravity of a symmetrical object will always fall on the axis of symmetry?

Answer. If the object is symmetrical, every subpart on the positive side of the axis of symmetry will be balanced by an identical part on the negative side. The first moment for the entire shape about the axis will sum to zero, meaning that

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} = \frac{0}{W} = 0.$$